Urban Economics and Analysis

Part of P. v. Mouche

Exercises A; 2024-2025

Exercise 1 *Consider the production function $f(k) = k^\alpha$ where $\alpha > 0$. So there is only one production factor k and thus the the input also is k. The production factor price is w (supposed to be positive).*

a. *Show the following:*

   $\alpha > 1 \Leftrightarrow$ *there is increasing returns to scale;*

   $\alpha = 1 \Leftrightarrow$ *there is constant returns to scale;*

   $\alpha < 1 \Leftrightarrow$ *there is decreasing returns to scale.*

b. *Determine the conditional production factor demand function $k^\star(q)$, i.e. the value of the cost minimising production factor k as a function of output q.*

c. *Let $C(q)$ be the cost function, i.e. the minimal costs in order to produce an output q. Show that $C(q) = wq^{\frac{1}{\alpha}}$.*

d. *Determine the average cost function $\mathrm{AC}(q) = C(q)/q$ and verify (the in Slides A mentioned result):*

$$\textit{increasing returns to scale} \Rightarrow \mathrm{AC} \textit{ is decreasing};$$

$$\textit{constant returns to scale} \Rightarrow \mathrm{AC} \textit{ is constant};$$

$$\textit{decreasing returns to scale} \Rightarrow \mathrm{AC} \textit{ is increasing}.$$

Exercise 2 *An important production function (and also utility function) is the Cobb-Douglas function*

$$f(k_1, k_2) = k_1^{\alpha_1} k_2^{\alpha_2};$$

here α_1 and α_2 are positive. In this exercise we explore its returning to scale properties and in doing so generalize Exercise 1a. (So here the input is (k_1, k_2).)

a. *Calculate $f(\lambda_1 k_1, \lambda k_2)$.*

b. Show that

$$\begin{aligned} \textit{increasing returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 > 1; \\ \textit{constant returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 = 1; \\ \textit{decreasing returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 < 1. \end{aligned}$$

Exercise 3 *Consider the problem of minimizing total transport costs for transporting over a given distance an input from a mine to a factory and for transporting the output to a market. We assume a 1-dimensional situation. The distance between market and mine is* 125 *km. The optimal location is the location of the factory where the sum of the transport costs of a given input and the to this input belonging output is minimal.*

Suppose the transport cost function that for the output (per ton) is $O(k) = \sqrt{k}$ euro and that for the input is $I(k) = \frac{1}{2}\sqrt{k}$ euro; here k is the distance in km.

a. Why these cost functions may be not so realistic?

b. Is this a weight gaining industry or weight losing industry?

c. Determine the total transport costs $T(k_0)$ if the factory is located at distance k_0 from the market.

d. Determine with a calculation the optimal distance k_0 and also sketch the graph of T.

e. Why, in fact, is it not necessary to determine the optimal k_0 by a calculation?

Exercise 4 *Make Exercis*

Exercise 5 *Consider the urban model from Chapter 2 in the text book for the residents utility function of Cobb-Douglas type $u(q, c) = qc$.*

a. Determine the formulas for the optimal floor space $q^\star(x)$ and bread consumption $c^\star(x)$ in terms of the income y, distance to the center x, rental price $p(x)$ and commuting costs t.

b. We know (by the equilibrium principle) that in the equilibrium the utility is independent of x; say this utility is w. Show that for the equilibrium rental price $p^\star(x)$ (i.e. the housing price curve) the equality $(y - tx)^2 = 4wp^\star(x)$ holds.

c. Show that $\frac{dp^\star}{dx}(x) = -\frac{t}{q^\star(x)}$.

d. Show that $p^\star(x) = \frac{p^\star(0)}{y^2}(y - tx)^2$.

e. Can we say something about the exact value of $p^\star(0)$ in part d?

f. Show that total equilibrium rent $p^\star(x) \cdot q^\star(x)$ is a decreasing function of x. Is this realistic?

g. *Now suppose the utility function is the Cobb-Douglas utility function* $u(q,c) = q^{\alpha_1} c^{\alpha_2}$ *with* $\alpha_1 + \alpha_2 = 1$ *and show that the formula in c continues to hold. (In fact this formula holds for a large class of "nice" utility functions.) Is the total equilibrium rent still a decreasing function of* x*?*

e 1.2 from the Text Book.

Exercise 6 *Consider the model for the Urban Spatial Structure (Chapter 2, 3 (and 4) in Textbook).* x *denotes the radial distance to the Central Business District. We suppose one income group. Are the following statements false or true?*

a. *Rental price* p *is a decreasing function of* x*.*

b. *Floor space* q *is an increasing function of* x*.*

c. *Amount of bread* c *is a decreasing function of* x*.*

d. *Price of land* r *is a decreasing function of* x*.*

e. *Building height* h *is a decreasing function of* x*.*

f. *Population density* D *is a decreasing function of* x*.*

Exercise 7 *Make Exercise 5.1 from the Text Book.*