Urban Economics and Analysis

Part of P. v. Mouche

Exercises A; 2023-2224

Exercise 1 *Consider the production function $f(k) = k^\alpha$ where $\alpha > 0$. So there is only one production factor k and thus the the input also is k. The production factor price is w (supposed to be positive).*

a. *Show the following:*

   $\alpha > 1 \Leftrightarrow$ *there is increasing returns to scale;*

   $\alpha = 1 \Leftrightarrow$ *there is constant returns to scale;*

   $\alpha < 1 \Leftrightarrow$ *there is decreasing returns to scale.*

b. *Determine the conditional production factor demand function $k^\star(q)$, i.e. the value of the cost minimising production factor k as a function of output q.*

c. *Let $C(q)$ be the cost function, i.e. the minimal costs in order to produce an output q. Show that $C(q) = wq^{\frac{1}{\alpha}}$.*

d. *Determine the average cost function $\mathrm{AC}(q) = C(q)/q$ and verify (the in Slides A mentioned result):*

$$\textit{increasing returns to scale} \Rightarrow \mathrm{AC} \textit{ is decreasing};$$

$$\textit{constant returns to scale} \Rightarrow \mathrm{AC} \textit{ is constant};$$

$$\textit{decreasing returns to scale} \Rightarrow \mathrm{AC} \textit{ is increasing}.$$

Exercise 2 *(Optional.) An important production function (and also utility function) is the Cobb-Douglas function*

$$f(k_1, k_2) = k_1^{\alpha_1} k_2^{\alpha_2};$$

here α_1 and α_2 are positive. In this exercise we explore its returning to scale properties and in doing so generalize Exercise 1a. (So here the input is (k_1, k_2).)

a. *Calculate $f(\lambda_1 k_1, \lambda k_2)$.*

b. *Show that*

$$\begin{aligned} \textit{increasing returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 > 1; \\ \textit{constant returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 = 1; \\ \textit{decreasing returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 < 1. \end{aligned}$$

Exercise 3 *Consider the problem of minimizing total transport costs for transporting over a given distance an input from a mine to a factory and for transporting the output to a market. We assume a 1-dimensional situation. The distance between market and mine is 125 km. The optimal location is the location of the factory where the sum of the transport costs of a given input and the to this input belonging output is minimal.*

Suppose the transport cost function (per ton) for the input is $I(k) = \frac{1}{2}\sqrt{k}$ *euro and that for the output (per ton) is* $O(k) = \sqrt{k}$ *euro; here* k *is the distance in km.*

a. *Why these cost functions may be not so realistic?*

b. *Is this a weight gaining industry or weight losing industry?*

c. *Determine the total transport costs* $T(k_0)$ *if the factory is located at distance* k_0 *from the market.*

d. *Determine with a calculation the optimal distance* k_0 *and also sketch the graph of* T.

e. *Why, in fact, is it not necessary to determine the optimal* k_0 *by a calculation?*

Exercise 4 *Make exercise 1.2 from the Text Book.*

Exercise 5 *Consider the model for the Urban Spatial Structure (Chapter 2, 3 (and 4) in Textbook).* x *denotes the radial distance to the Central Business District. We suppose one income group. Are the following statements false or true?*

a. *Rental price* p *is a decreasing function of* x.

b. *Floor space* q *is an increasing function of* x.

c. *Amount of bread* c *is a decreasing function of* x.

d. *Price of land* r *is a decreasing function of* x.

e. *Building height* h *is a decreasing function of* x.

f. *Population density* D *is a decreasing function of* x.

Exercise 6 *Make Exercise 5.1 from the Text Book.*

Short solutions.

Solution 1 a. $f(\lambda k) = (\lambda k)^\alpha = \lambda^\alpha f(k) \begin{cases} > \lambda f(k) \text{ if } \alpha > 1, \\ = \lambda f(k) \text{ if } \alpha = 1, \\ < \lambda f(k) \text{ if } \alpha < 1. \end{cases}$

b. If the producer wants to produce an amount q with minimal costs, then the input k has to satisfy $k^\alpha = q$. Therefore $k^\star(q) = q^{1/\alpha}$.

c. $C(q) = wk^\star(q)$. Thus, by part b, $C(q) = wq^{\frac{1}{\alpha}}$.

d. For the average costs $\mathrm{AC}(q)$ we obtain

$$\mathrm{AC}(q) = \frac{\mathrm{C}(q)}{q} = wq^{\frac{1}{\alpha}-1} = wq^{\frac{1-\alpha}{\alpha}}.$$

e. By the formula in d. For example, for $\alpha = 1/2$ we have decreasing returns to scale and $\mathrm{AC}(q) = wq$. So $AC(q)$ is increasing.

Solution 2 1. $f(\lambda_1 k_1, \lambda k_2) = (\lambda k_1)^{\alpha_1} (\lambda k_2)^{\alpha_2} = \lambda^{\alpha_1} k_1^{\alpha_1} \lambda^{\alpha_2} k_2^{\alpha_2} = \lambda^{\alpha_1+\alpha_2} k_1^{\alpha_1} k_2^{\alpha_2} = \lambda^{\alpha_1+\alpha_2} f(k_1, k_2)$

2. By part 1 (as $\lambda > 0$).

Solution 3 a. As $I(0) = O(0) = 0$, there are no terminal costs.

b. Weight gaining industry as $O(k) > I(k)$ (for positive k).

c. $T(k_0) = O(k_0) + I(125 - k_0) = \sqrt{k_0} + \frac{1}{2}\sqrt{125 - k_0}$.

d. Derivative $T'(k_0) = \frac{1}{2\sqrt{k_0}} - \frac{1}{2}\frac{1}{2\sqrt{125-k_0}} = \frac{1}{2}(\frac{1}{\sqrt{k_0}} - \frac{1}{2\sqrt{125-k_0}})$.

We see: derivative zero if $\sqrt{k_0} = 2\sqrt{125 - k_0}$. I.e. $k_0 = 4(125 - k_0)$. We find $k_0 = 100$.

Also derivative is positive if $k_0 < 100$ and negative if $k_0 > 100$. This implies $k_0 = 0$ or $k_0 = 125$ is a minimiser. As $T(125) = \sqrt{125} > \frac{1}{2}\sqrt{125} = T(0)$, 0 is the minimiser.

d. Optimal to locate at the market.

e. Not necessary as we as theory also predicts this result: we have to do with a weight gaining industry and there are economics of distance (as average transportation costs for input and for outputs are decreasing).

Solution 4 a. One factory in city D gives minimises the costs.

b. One factory in each of the four cities minimises the costs.

c. Scale economies are weaker in part b compared to part a.

d. $t = 4$.

Solution 5 All are true.

Solution 6 a. Mr. 1 and Mr. 2 on freeway; Mr. 3 on alternate route.

b. First: $7 + 5 = 3 = 15$. Second: $2 + 5 + 3 = 10$. Third: $10 + 3 = 13$. Fourth: $9 + 9 + 9 = 27$.

c.Optimal: Mr. 1 on freeway; Mr. 2 and Mr. 3 on alternate route.