Game Theory:
Cooperative Games

Hans-Peter Weikard
Enivironmental Economics and Natural Resources (ENR), room 1118, Lebo

WAGENINGEN UR
For quality of life

O. Morgenstern, J. v. Neumann, L. Shapley, J. Nash

Cooperative Games

Plan for today

Cooperative vs. non-cooperative games

Axiomatic method

Two cooperative solution concepts

- Core
- Shapley Value

Friday

Bargaining games and bargaining solutions

Cooperative Games

		Player 2	
		left	right
Player 1	up	2, 2	-1, 3
	down	3, -1	0, 0

- Individual rationality
- Collective rationality

In a cooperative game players can make binding agreements

Consequently, (Pareto) efficient outcomes can be achieved.

Cooperative Games

Non-cooperative	Cooperative
Individual rationality	Individual and collective rationality
Sequence of choice can matter, game trees	Sequence of choice is irrelevant, no tree structure
Behavioural strategies matter	Behavioural strategies are implicit or absent
Solution found by assessing strategic choices	Solution found by assessing characteristics of payoffs, i.e., axiomatic method
Solution concept: Nash equilibrium and refinements	Many different solution concepts
Frequently inefficient solutions	Efficiency usually guaranteed
Usually meant to explain or predict outcomes	Usually meant as a normative approach

Axiomatic approach (1)

Axioms are postulates taken to be true
(also called “first principles” or “premises”).

They form the bases of deductive systems.

Major developments around 1900 with the rise of mathematical logic.

Ideas go back Euclid (300 BC).

Axiomatic approach (2)

Example: Peano's axioms of number theory

- 0 is a natural number.
- For every natural number x, $x = x$ (reflexivity)
- For all natural numbers x, y, if $x = y$, then $y = x$ (symmetry)
- For all natural numbers x, y, z if $x = y$ and $y = z$, then $x = z$ (transitivity)
- ...
- Every natural number has a unique successor.
- …

Axiomatic approach in cooperative game theory

Example: Axiomatic bargaining or cost sharing

e.g.,

- Anonymity
- Symmetry
- Monotonicity

Solutions are characterised by their properties.

Preliminaries: Notation in set theory

Set of players	$N = \{1, 2, \ldots, i, \ldots, n\}$
Subset (coalition)	$S \subseteq N$
Union	$S \cup T$
Intersection	$S \cap T$
Empty set	$\varnothing$
S without *T*	$S \setminus T$
Complement	$N \setminus S$
Power set	$\mathrm{P}(N)$

Preliminaries

TU games (transferable utility games), utility is linear in money.

Coalitional games

We have a set of players *N*.

Subsets of players (coalitions) are called $S \subseteq N$.

Payoffs are defined for coalitions.

We call $v(S)$ the worth of the coalition.

Individual payoffs x_i must satisfy $\sum_{i \in S} x_i \leq v(S)$.

A game is a pair (N, v),

e.g., market games, cost sharing games, voting games.

Core of TU games

(1) $v(\varnothing) = 0$

(2) $v(S \cup T) \geq v(S) + v(T)$; for $S \cap T = \varnothing$

(1) is a normalisation.

(2) is the super-additivity condition

$v(S)$ is the payoff that a coalition can insure for itself; the maximin value.

(3) $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$

Condition (3) defines a convex game, but also

(4) $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for $S \subset T$.

Core of TU games

An imputation is a payoff vector *x* that distributes the grand coalition payoff between players satisfying individual rationality and Pareto optimality.

$$\sum_{i \in N} x_i = v(N); \text{ and for all } i, \; x_i \geq v(\{i\}).$$

Domination: *x* dominates *x*' in *S* if for all $i \in S$, $x_i \geq x_i'$ and the inequality is strict for some $i \in S$.

Core of TU games

The core is the set of all undominated imputations.

For an imputation (payoff vector) x in the core it must hold that there is no $S \subset N$ such that

$$v(S) > \sum_{i \in S} x_i$$

Thus, an imputation in the core is individually and collectively rational.

The core of game Γ is a set

$$C(\Gamma) = \{x : v(S) - \sum_{i \in S} x_i \leq 0, \text{ for all } S \subset N\}$$

Core of TU games

$$C(\Gamma) = \{x : v(S) - \sum_{i \in S} x_i \leq 0, \text{ for all } S \subset N\}$$

No solution in the core can be blocked by any coalition.

Core of TU games

Example: Water provision for municipalities.

Four cities $N = \{1, 2, 3, 4\}$ have to be provided with water. Each has variable cost of water of 100. The construction of a well costs 200. Any well is sufficient to supply water for all. Cities 1 and 2 are in the West, 3 and 4 are in the East. West cities can share a well, so can East cities. East and West can be connected through a pipeline that costs 100.

Construct the characteristic function.

The core with ordinal preferences (non-TU)

- *An undesired guest (see Bogomolnaia and Jackson GEB 2002)*
- Let N = {1, 2, 3} and
- 1’s preference order: {1, 2} ,{1}, {1, 2, 3}, {1,3}
- 2’s preference order: {1, 2} ,{2}, {1, 2, 3}, {2,3}
- 3’s preference order: {1, 2, 3},{ 2, 3}, {1, 3}, {3}
- These orderings can be represented by additively separable utilities. Here,
- {1 ,2}, {3} is in the core and is individually stable.

The core with ordinal preferences (non TU)

Two is company, three is a crowd. *(see Bogomolnaia and Jackson GEB 2002)*

Let N = 1 2 3 and

- 1’s preference order: {1, 2}, {1, 3}, {1, 2, 3}, {1}
- 2’s preference order: {2,3}, {2, 1}, {1, 2, 3}, {2}
- 3’s preference order: {3, 1}, {3, 2}, {1, 2, 3}, {3}
- These preferences have a cycle: the first player prefers the second player to the third. The second player prefers the third player to the first. The third player prefers the first one to the second.
- All players prefer to be in some couple over being all together, and being alone is the worst outcome.
- Here, the core is empty.

Shapley Value

For a coalitional game with characteristic function $v(S)$, the Shapley value assigns to each player $i \in N$

$$\varphi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{(s-1)!(n-s)!}{n!} \left(v(S) - v(S \setminus \{i\}) \right)$$

where s and n are the numbers of the members of S and N, respectively. We have

$$\sum_{i \in S} \varphi_i(v) = v(N).$$

The Shapley value is the average expected contribution of one player considering all possible coalitions to which a player can contribute.

Shapley Value

The Shapley value is the *unique* imputation that satisfies

- Group rationality (efficiency)
- Symmetry (the order of players does not matter)
- Additivity $\varphi_i(v+w) = \varphi_i(v) + \varphi_i(w)$

It also satisfies the Null player condition

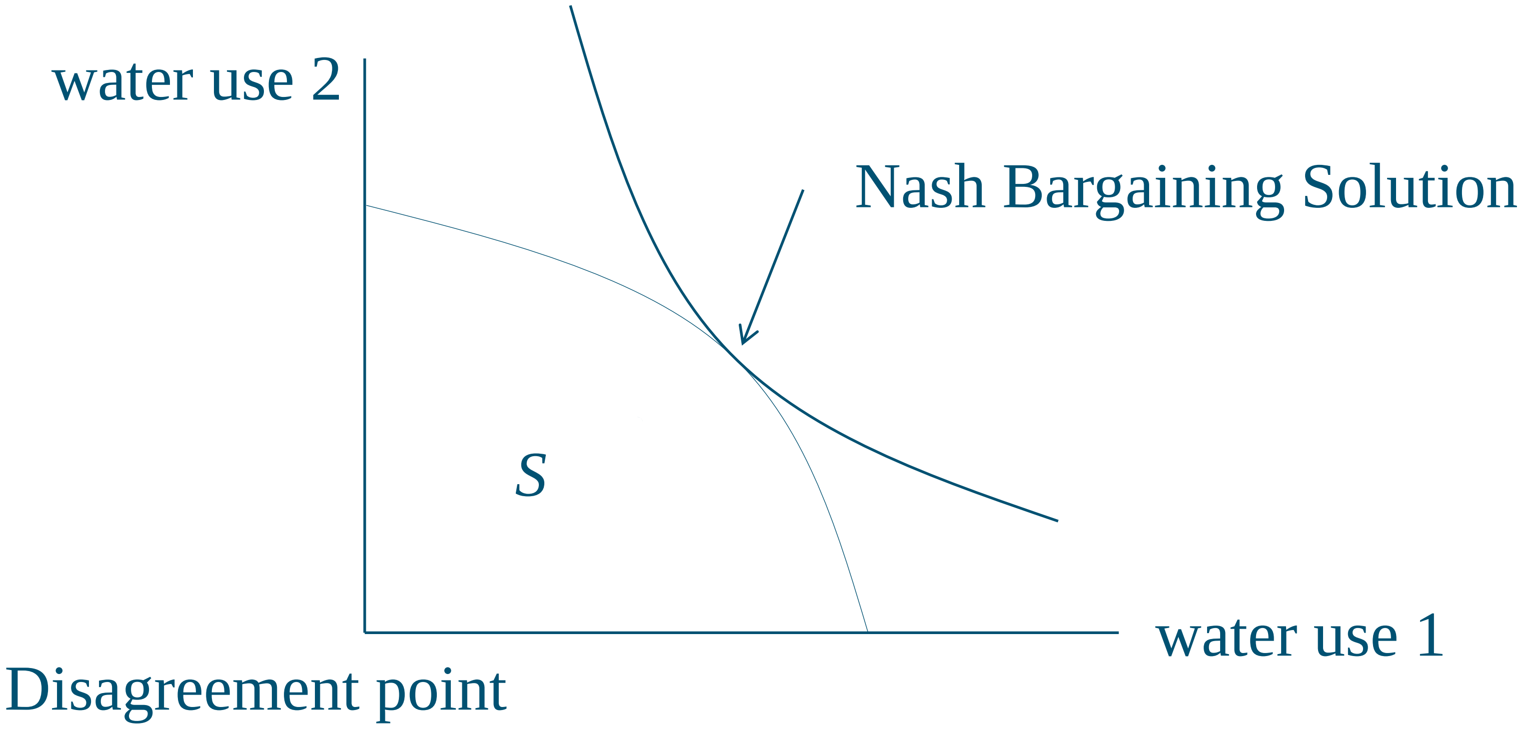
$$\varphi_i(v) = 0 \text{ if for all } S \;\; v(S \cup \{i\}) - v(S) = 0.$$

Bargaining

Plan for today

- Axiomatic bargaining
  - Nash Bargaining Solution
- Non-cooperative two-player bargaining
  - Rubinstein's bargaining game
- The Nash programme

Nash Bargaining Solution: Example



Nash Bargaining solution

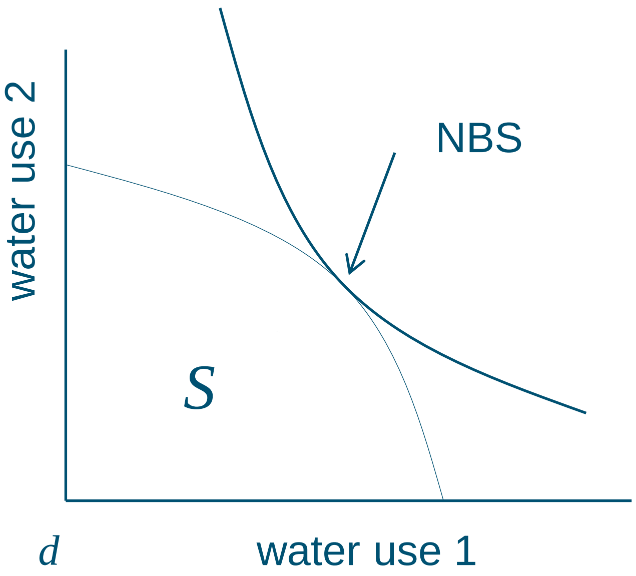
Nash Bargaining Solution

$$\max \underbrace{(u_1 - d_1)^{\alpha_1} \cdot (u_2 - d_2)^{\alpha_2}}_{\text{Nash product}}$$

α_1, α_2 are the bargaining weights

$$\sum_{i \in N} \alpha_i = 1$$

The solution satisfies (i) Invariance to Equivalent Utility Representations, (ii) Symmetry, (iii) Independence of Irrelevant Alternatives, and (iv) Pareto efficiency.



Nash Bargaining Solution

Nash Bargaining: Axioms

Bargaining problem: A set of possible outcomes and a threatpoint (S, d)

A solution should satisfy:

A1: Independence of utility transformations

A2: Symmetry

A3: Independence of irrelevant alternatives

A4: Pareto optimality

Nash, J. (1950) The Bargaining Problem. Econometrica 18, 286-295.

Roth, A. (1979) Axiomatic Models of Bargaining. Lecture Notes in Economics and Mathematical Systems 170. Berlin: Springer.

Bargaining: Alternative Solution concepts

Kalai – Smorodinski solution

Bargaining problem: A set of possible outcomes and a threatpoint (S, d)

A solution should satisfy:

A1: Independence of utility transformations

A2: Symmetry

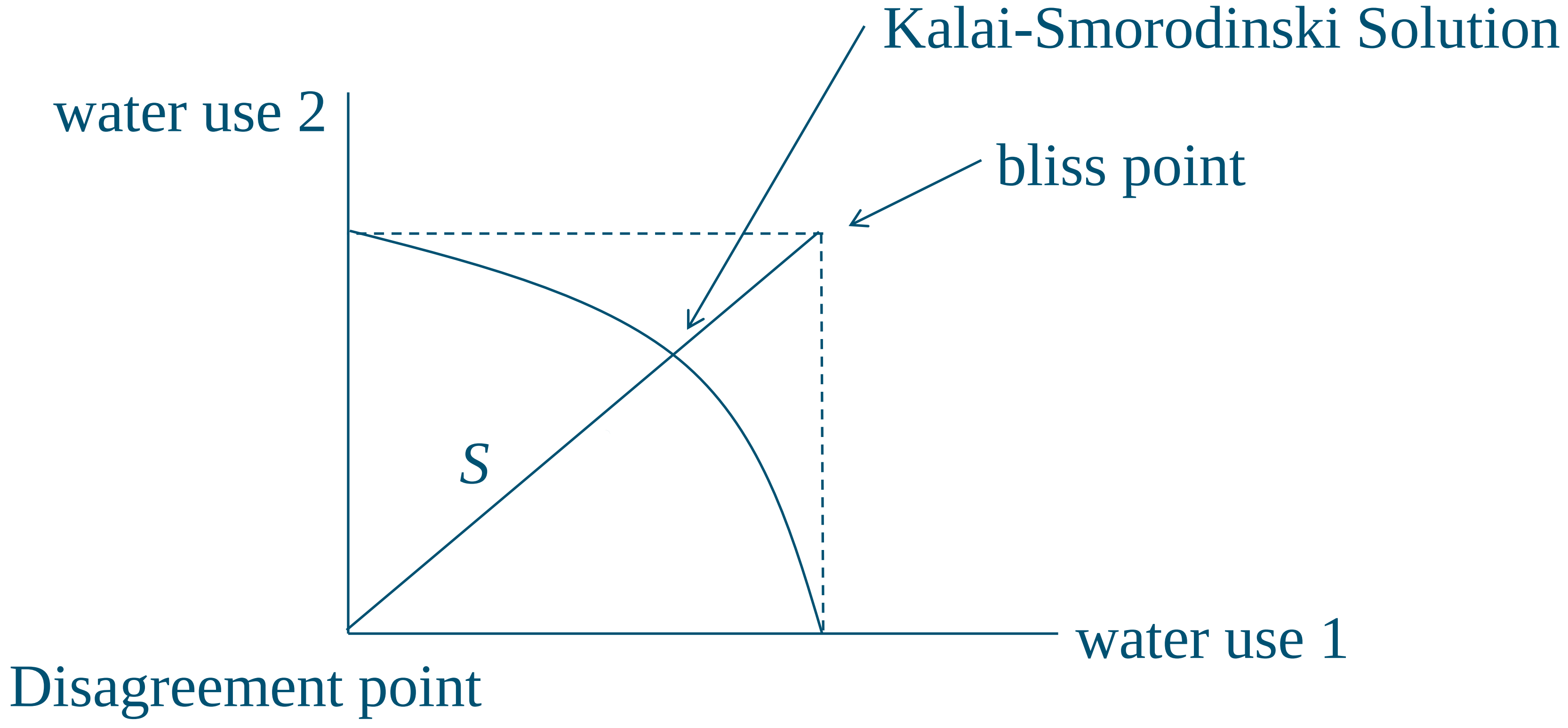
A3: Monotonicity

A4: Pareto optimality

Nash, J. (1950) The Bargaining Problem. Econometrica 18, 286-295.

Roth, A. (1979) Axiomatic Models of Bargaining. Lecture Notes in Economics and Mathematical Systems 170. Berlin: Springer.

Nash Bargaining Solution: Example



The Nash programme

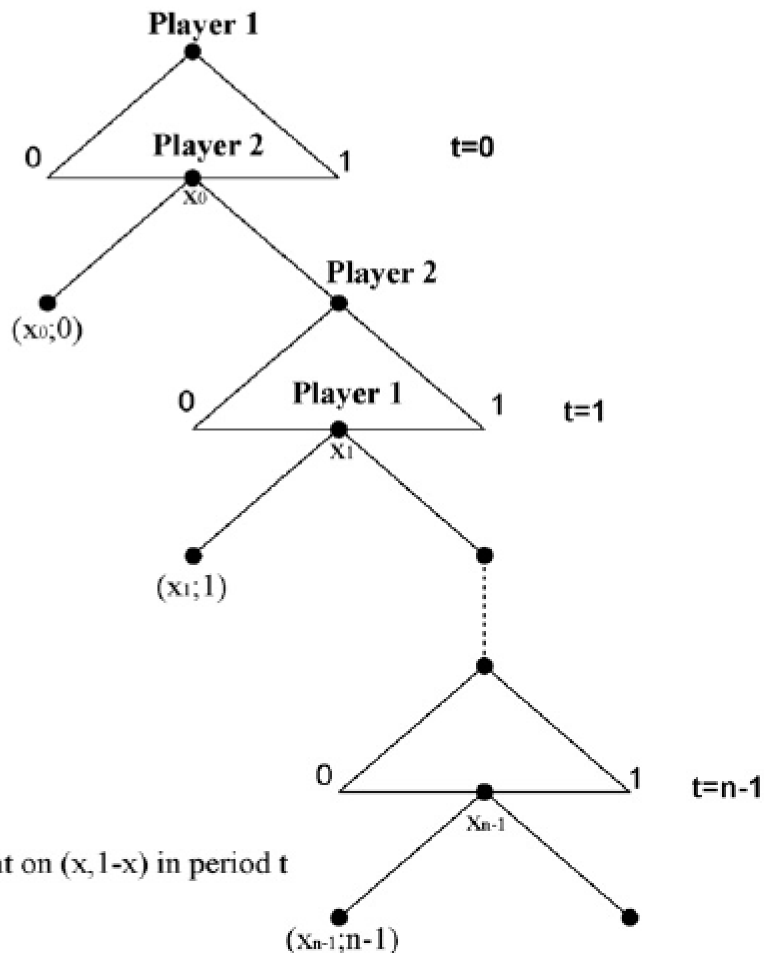
Rubinstein's bargaining game:

The alternating offer model:

Players' shares:

$$\pi_1 = \frac{1-\delta_2}{1-\delta_1\delta_2};$$

$$\pi_2 = 1 - \frac{1-\delta_2}{1-\delta_1\delta_2} = \frac{1-\delta_1}{1-\delta_1\delta_2}$$



(x;t) : agreement on (x,1-x) in period t

Implements the NBS

Rubinstein, A. (1982) Perfect Equilibrium
in a Bargaining Model. Econometrica 50, 97-109.