Game Theory

Lesson 4: Games in Extensive Form

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What You will learn

After studying Lesson 4, You

- should know what we mean by a game in extensive form and know various notions for such a game;
- should be able to transform a game in extensive form into a game in strategic form by means of normalisation;
- should be familiar with the notion of value and optimal strategy of an antagonistic game;
- should know the Theorem of Kuhn;
- should know how to apply the things You have learned to concrete games like Tic-Tac-Toe, Hex and Nim.

Mathematical types of games

Up to now we developed theory for games in strategic form; in such games there one can say that each player makes a single move. Our concrete games Tic-Tac-Toe, Hex and Nim do have a more rich structure as there are more moves. Now we are going to develop a little bit theory for such games.

In fact there are at least three main (mathematical) types of games:

- Games in strategic form, with bimatrix games as a special case. (Lessons 2 and 3)
- Games in extensive form. (Lesson 4, i.e. this lesson)
- Games in characteristic function form. (Third week.)

Game in extensive form

The setting in the remainder of this lesson always is a non-cooperative one with complete information and perfect information and no chance moves.

A formal definition of a game in extensive form is quite technical. Below follows a more or less ‘visual’ definition.

Game in extensive form (ctd.)

A game in extensive form can be represented by a game tree. In such a tree

- there are nodes (also called histories): end nodes, decision nodes and a unique initial node;
- there are a (directed) branches;
- there are payoffs at the end nodes;
- each non-initial node has exactly one predecessor.

We further always assume that the game is finite, i.e. that the number of nodes and branches is finite.

Notions for games in extensive form

An outcome of the game is a path through the game tree, or equivalently a terminal node of the game tree.

Given a game in extensive form, for a player the notion of completely elaborated plan of playing is defined. One may consider it as a piece of paper on which that player writes which move he will make if it is his turn. If each player does this, the game can be played and an outcome results.

As already said, a primary purpose of game theory is to determine the outcomes of games according to a solution concept. For us in this simple Game Theory course the concept is that of Nash equilibrium.

Notion of strategy

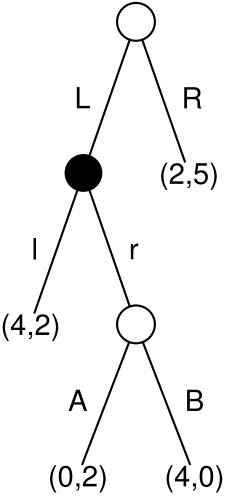
A related notion is that of strategy of a player: this is specification at each decision node how to move.

Note that a strategy may be much more than a completely elaborated plan of play: the player also has to specify his moves at nodes which never may be reached. This sounds strange. This notion of strategy in fact is very important in order to understand how games will be played.

However, in this simple Game Theory course we shall not further explore this. The topic involved here is that of subgame perfect Nash equilibrium. Please see the little book of K. Binmore for more.

Example

Here is an example of a game tree:



Example (ctd.)

In this game tree, the nodes where player 1 moves are presented by the $\circ$ symbol and those for player 2 by the $\bullet$ symbols.

The possible moves are denoted by the symbols L, R, l, r, A, B.

The payoffs are at given at the end nodes (for which no symbol is used).

Example (ctd.)

In this game

- Player 1 has 4 strategies: *LA*, *LB*, *RA* and *RB*.
- Player 2 has 2 strategies: *l* and *r*.
- Playing *R* is a completely elaborated plan of play for player 1, but it is not a strategy. If You do not understand this, then look back to the definition of strategy!

This game will be dealt with further in Exercises 3.

Normalisation

Out of a given a game in extensive form, one can make in a natural way a game in strategic form: just consider for each player all possible strategies and calculate the payoffs at each possible strategy profile.

This transformation of a game in extensive form into a game in strategic form is called normalisation .

As a game in strategic form is a game with imperfect information, normalisation destroys the perfect information.

Normalisation makes that all terminology and results for games in strategic form now also applies to games in extensive forms. In particular the notion of Nash equilibrium.

Normalisation (ctd.)

In particular the chess game can be represented as a game in strategic form. As this is a finite antagonistic game, even by a bimatrix game.

Such a bimatrix is really huge and has as entries only 1's, 0's and -1's.

Value

In Lesson 3 we have proven that in an antagonistic game the following holds for each player *i*: the payoff of player *i* is in each Nash equilibrium the same. This makes that the following notion of value is well-defined:

Given an antagonistic game in extensive form, its value is defined as the payoff of player 1 in a Nash equilibrium.

A very important question know is whether a Nash equilibrium exists, implying that the game has a value. (If a Nash equilibrium does not exist, the game does not have a value.) In this context a theorem of Kuhn is important.

Theorem of Kuhn

Theorem

(Kuhn) Each finite game in extensive form with perfect information has a Nash equilibrium.

Unfortunately i do not have enough time to prove it, but i shall give now some intuition.

Well, the idea to find Nash equilibria of a game in extensive form (and in particular the value of an antagonistic game is to analyse it 'from its end to its beginning').

I now illustrate this with the following game (without drawing the game tree).

Solving from the end to the beginning

There is a pillow with 100 matches. They alternately remove $1, 3$ or 4 matches from it. (Player 1 begins.) The player who makes the last move wins.

How this game will be played?

Solving from the end to the beginning (ctd.)

If there are 0 matches left, then the player who has to play loses (as in fact the game is already over). So 0 is a losing position.

If there is 1 match left, then the player who has to play wins. So 1 is a winning position.

If there are 2 matches left, then the player who has to play has to remove one match and a position which 1 match remains. As this is a winning position, 2 is a losing position.

If there are 3 matches left, then the player who has to play can remove these matches and then wins. So 3 is a winning position.

Solving from the end to the beginning (ctd.)

If there are 4 matches left, then the player who has to play can remove these matches and then wins. So 4 is a winning position.

If there are 5 matches left, then the player who has to play can remove 3 matches and a position which 2 matches remains. As this is a losing position, 5 is a winning position.

If there are 6 matches left, then the player who has to play can remove 4 matches and a position which 2 matches remains. As this is a losing position, 6 is a winning position.

If there are 7 matches left, then the player who has to play always will end up in a winning position. So 7 is a losing position.

Solving from the end to the beginning (ctd.)

And so on: the loosing positions are $0, 2, 7, 9, 14, 16, 21, \ldots$, i.e. the numbers that have remainder 0 or 2 when divided by 7.

Because $100/7$ has remainder 2, 100 is a loosing position. Conclusion: player 2 always can win the game, implying that he has a winning strategy.

Conclusion: game has value -1. And from the above it is clear that player 2 knows how to win, i.e. has an optimal ‘play’.

Optimal strategy

More formally: given an antagonistic game with a value, an optimal strategy for a player is a completely elaborated plan of playing for this player that guarantees this player at least the value.

Values

Here are results for the value of some concrete games:

	tic-t.-t.	chess	checkers	hex	nim
value	draw	not known	draw	1 wins	known
opt. strat.	known	not known	known	not known	known

It is interesting to note that all these games have a value (thanks to Kuhn's theorem), but that the actual value up to now for chess is not known. Also interesting to note is that we know that player 1 can win each hex game, but that we do not know how he could do this.

Predictions

With the theory in this lesson we have seen that we can make very strong predictions how rational intelligent players will play antagonistic games. The reason is that all these games game have a value which will be realized if each player plays optimal.

If the game is not antagonistic, then we also can make predictions by means of Nash equilibria, however the quality of such predictions is less than in the case of antagonistic games.

Hex-game revisited

Finally, we show here that in the Hex-game player 2 cannot have a winning completely elaborated plan of playing. This we do by showing that if player 2 would have such a plan, player 1 also would have one, which is absurd.

The very clever proof that is presented here is a so-called ‘strategy stealing argument’. (Probably You have to read this proof very slowly and probably many times in order to understand it.)

Hex-game revisited (ctd.)

So suppose that player 2 has a winning completely elaborated plan of playing, which we will call *S*. Now consider the following completely elaborated plan of playing for player 1.

Player 1 makes his first move at random. Thereafter he should pretend to be player 2, 'stealing' the second player's elaborated plan of playing *S*, and follow strategy *S*, which by hypothesis will result in a victory for him. What, of course, we still have to show is that this is a legitimate completely elaborated plan of playing. Well, if strategy *S* calls for him to move in the hexagon that he chose at random for his first move, he should choose at random again. This will not interfere with the execution of *S*.

Finally, note that the above strategy of player 1 is at least as good as strategy *S* of player 2 since having an extra marked square on the board is never a disadvantage in hex.