Game Theory

Lesson 2: Bimatrix Games

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What You will learn

After studying Lesson 2, You should be able to determine for a bimatrix game

- whether the game is an antagonistic game;
- the strictly dominant strategies;
- the strictly dominant equilibria;
- the Nash equilibria;
- the weakly Pareto efficient strategy profiles;
- the fully cooperative strategy profiles;
- whether the game is a prisoners' dilemma.

These notions are very fundamental. In addition:

- You should be familiar with the specific game, i.e. the Hotelling Game, introduced in this lesson.

What You will learn (ctd.)

We start our theory with the notion of bimatrix game and introduce the above already mentioned very fundamental notions for such a game.

After a training with such notions, we apply them (if possible) to new concrete games.

Bimatrix game

So what is a bimatrix game ? Well, bimatrix games concern the most simple type of so-called games in strategic form dealing with two players; player 1 and player 2. The game is represented by a so-called bimatrix (which explains its name). For example:

$$\begin{pmatrix} 3;3 & 2;2 \\ 7;-1 & -3;1 \\ 1;2 & 12;-9 \end{pmatrix}.$$

Bimatrix game (ctd)

- This is a 3×2-bimatrix game, i.e. it has 3 rows and 2 columns.
- Player 1 chooses a row: row 1, row 2 or row 3, meaning that player 1 has 3 strategies. Player 2 chooses a column: column 1 or column 2, meaning that player 2 has 2 strategies. These choices are made simultaneously and independently. We refer to a choice of a player as strategy and to a choice of both players as strategy profile .

Bimatrix game (ctd.)

- In each of the cells of the bimatrix there is a pair of numbers, separated by a semicolon. These numbers represent the payoffs; the first number concerns player 1 and the second player 2.
- For example: at the strategy profile $(3, 2)$, i.e. row 3 and column 2, player 1 has payoff 12 and player 2 has payoff -9.

Bimatrix game: concrete examples

Many games can be represented in a natural way as a bimatrix game. For example stone-paper-scissors:

$$\begin{pmatrix} 0;0 & -1;1 & 1;-1 \\ 1;-1 & 0;0 & -1;1 \\ -1;1 & 1;-1 & 0;0 \end{pmatrix}$$

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Indeed: first strategy is stone, second paper and third scissors. If players make the same choice, then it is draw: payoffs 0 for both. If players make a different choice, then there is a winner with payoff 1 and a looser with payoff -1.

Hotelling Game

Here we consider another example of an economics game: a discrete variant of the original Hotelling Game.

The (Discrete) Hotelling Game depends on a parameter m being a positive integer. Consider the $m + 1$ points of $H := \{0, 1, \ldots, n\}$ on the real line, to be referred to as *vertices*.

Two players simultaneously and independently choose a vertex. If player 1 (2) chooses vertex x_1 (x_2), then the payoff $f_i(x_1, x_2)$ of player i is the number of vertices that is the closest to his choice x_i; however, a shared vertex, i.e. one that has equal distance to both players, contributes only $1/2$.

Hotelling Game (ctd.)

Various (economic) interpretations of this game are possible: can You provide one?

Answer:

Hotelling Game (ctd.)

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Answer: Imagine a stretch of beach on which two ice cream retailers want to sell ice cream. The flavours they offer and the prices they charge are the same, so sunbathers go to the closest cart. The question for the two retailers is, where should they set up their carts to get the most customers? In fact there are various variants of this model. Above it is assumed that there is a finite number (i.e. $m+1$) positions where the sunbathers can enjoy there live.

Hotelling Game (ctd).)

Example $m = 7$.

Strategy profile (5,2) :

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Payoffs:

$1 + 1 + 1 + 1 = 4$

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Hotelling Game (ctd).)

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Strategy profile (5,2) :

Payoffs:

$1 + 1 + 1 + 1 = 4$

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Strategy profile (0,3) :

Payoffs

$1 + 1 = 2$

$1 + 1 + 1 + 1 + 1 + 1 = 6$

Hotelling Game (ctd.)

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Hotelling Game (ctd.)

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Strategy profile $(2, 6)$:

Payoffs:

$1 + 1 + 1 + 1 + \frac{1}{2} = 4\frac{1}{2}$

$\frac{1}{2} + 1 + 1 + 1 = 3\frac{1}{2}$

Hotelling Game (ctd.)

Example $m = 7$.

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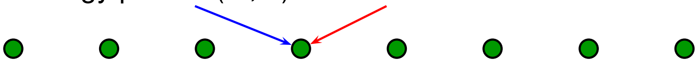
Strategy profile (3,3) :



Hotelling Game (ctd.)

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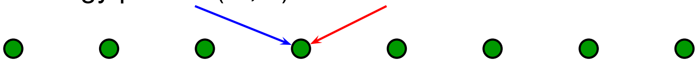
Strategy profile (3,3) :



Payoffs:

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4$

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Making predictions

Knowing what a bimatrix game is, we can now introduce some notions that are useful for making predictions about reasonable outcomes of such a game.

But first a motivating example:

How to play this game?

Consider the following game.

$$\begin{pmatrix} -1; -1 & -3; 0 \\ 0; -3 & -2; -2 \end{pmatrix}.$$

What would You as (a rational intelligent) player 1 play in this game: row 1 or row 2?

How to play this game? (ctd.)

May be Your answer is row 2. And Your opponent may answer column 2 as these choices are the best one can do. So the result then will be a payoff of -2 for You both. Is this a rational outcome (whatever this means)? Note that playing row 1 and column 1 would be better for both of You.

Note: this game is the classical prisoners' dilemma game (of Tucker) where the payoffs correspond to years in prison.

Fundamental notions

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- Strictly dominant strategy of a player: the best strategy of that player independently of the strategies of the other players.
- Strictly dominant equilibrium : strategy profile in which each strategy is strictly dominant.
- Nash equilibrium : strategy profile with the property that no player regrets his choice.

Fundamental notions (ctd.)

- A strategy profile **b** is an unanimous Pareto improvement of a strategy profile **a** if each player has in **b** a greater payoff than in **a**.
- A strategy profile **x** is weakly Pareto efficient if there does not exist an unanimous Pareto improvement of **x**.
- A strategy profile **x** is weakly Pareto inefficient if there exists an unanimous Pareto improvement of **x**.

Fundamental notions (ctd.)

A strategy profile is

- fully cooperative if the total payoff in this strategy profile is maximal.

A prisoners' dilemma game is a game with a strictly dominant equilibrium that is weakly Pareto inefficient.

Finally: an antagonistic game is a game were the total payoff is zero in each strategy profile.

Pareto efficiency

So one also can say: a strategy profile is weakly Pareto efficient if there is no other strategy profile in which each player is better off.

Remark: there is another notion of Pareto efficiency, also called Strong Pareto efficiency . In fact, this notion is the usual efficiency notion in economics. In words: a strategy profile is strongly Pareto efficient if there is no other strategy profile in which at least one player is better of and no player is worse off. Be happy: we shall not consider this (for many students quite difficult) notion further in our simple Game Theory course!

Pareto-efficiency is one of the most important notion in economics.

Solution concepts

The aim of game theory is to understand/predict how games will be played. Here so-called solution concepts play a role. For games in strategic form the following one are important: Strictly dominant equilibrium and Nash equilibrium.

In our simple Game Theory course we do not have time to consider other solution concepts for games in strategic form.

Youtube videos

There are a lot of Youtube videos dealing with the above topics. For example: the following video.

`https://www.youtube.com/watch?v=pC--lK8KNwo` (for what we did up to now the video is relevant for period 0:00-7:20 and period 9:05-14:20).

Concerning this video:
1. What is called 'one-shot game' in the video we have called 'bimatrix game'.
2. Please forget the terminology 'normal form game' in the video (we shall deal later with it).
3. What is called 'dominant strategy' in the video, we have called 'strictly dominant strategy'.

Examples

1. Determine the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 2;4 & 1;4 & 4;3 & 3;0 \\ 1;1 & 1;2 & 5;2 & 6;1 \\ 1;2 & 0;5 & 3;4 & 7;3 \\ 0;6 & 0;4 & 3;4 & 1;5 \end{pmatrix}.$$

Examples

1. Determine the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 2;4 & 1;4 & 4;3 & 3;0 \\ 1;1 & 1;2 & 5;2 & 6;1 \\ 1;2 & 0;5 & 3;4 & 7;3 \\ 0;6 & 0;4 & 3;4 & 1;5 \end{pmatrix}.$$

No strictly dominant strategies. Nash equilibria: strategy profiles $(1,1), (1,2), (2,2)$ and $(2,3)$.
Attention: a notation as $(2,3)$ here above denotes the strategy profile where player 1 plays row 2 and player 2 plays column 3. So it deals with strategies and not with payoffs (which in strategy profile $(2,3)$ are 5 for player 1 and 2 for player 2).

Examples (ctd.)

2. Determine the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 6;1 & 7;1 & 6;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}$$

Examples (ctd.)

2. Determine the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 6;1 & 7;1 & 6;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}$$

Strictly dominant strategies for player 1: strategy 1. Strictly dominant strategies for player 2: none. Nash equilibria: strategy profile $(1,3)$.

Examples (ctd.)

3. Determine the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 3;-1 & 3;1 & 6;0 \\ 1;0 & 3;1 & 8;0 \\ 5;2 & 4;1 & 8;2 \end{pmatrix}.$$

Examples (ctd.)

3. Determine the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 3;-1 & 3;1 & 6;0 \\ 1;0 & 3;1 & 8;0 \\ 5;2 & 4;1 & 8;2 \end{pmatrix}.$$

No strictly dominant strategies. Nash equilibria: strategy profiles $(3,1)$ and $(3,3)$. For example: row 3 (i.e. strategy 3) of player 1 is not strictly dominant as it gives the same payoff (and not a greater payoff) as row 2 in case player 2 plays column 3.

Examples (ctd.)

4. Determine the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 1;0 & 1;2 & 0;4 \end{pmatrix}.$$

Examples (ctd.)

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$$\begin{pmatrix} 1;0 & 1;2 & 0;4 \end{pmatrix}.$$

   Strictly dominant strategies for player 1: strategy 1. Strictly dominant strategy for player 1: strategy 3. Nash equilibria: strategy profiles $(1,3)$.

Examples (ctd.)

5. Determine the weakly Pareto efficient strategy profiles for

$$\begin{pmatrix} 1;0 & 3;1 & 6;0 \\ 2;1 & 4;1 & 8;1 \end{pmatrix}.$$

Examples (ctd.)

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$$\begin{pmatrix} 1;0 & 3;1 & 6;0 \\ 2;1 & 4;1 & 8;1 \end{pmatrix}.$$

Weakly Pareto efficient strategy profiles: (1,2), (2,1), (2,2), (2,3).

Examples (ctd.)

6. Determine the weakly Pareto efficient strategy profiles for

$$\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}.$$

Examples (ctd.)

6. Determine the weakly Pareto efficient strategy profiles for

$$\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}.$$

Weakly Pareto efficient strategy profiles:
$(1,1),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$.

For example: $(1,2)$ is not weakly Pareto efficient as $(2,2)$ is an unanimous Pareto improvement of $(1,2)$.

Examples (ctd.)

7. Determine the fully cooperative strategy profiles for

$$\begin{pmatrix} 1;0 & 1;-4 & 0;1 \\ 1;1 & 0;2 & -2;0 \end{pmatrix}.$$

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$$\begin{pmatrix} 1;0 & 1;-4 & 0;1 \\ 1;1 & 0;2 & -2;0 \end{pmatrix}.$$

Fully cooperative strategy profiles: $(2,1),(2,2)$.

Examples (ctd.)

8. Determine the strictly dominant equilibria for the following game. Is the game a prisoners' dilemma?

$$\begin{pmatrix} 1;0 & -1;4 & 0;2 \\ 0;6 & 0;2 & 0;3 \end{pmatrix}.$$

Examples (ctd.)

8. Determine the strictly dominant equilibria for the following game. Is the game a prisoners' dilemma?

$$\begin{pmatrix} 1;0 & -1;4 & 0;2 \\ 0;6 & 0;2 & 0;3 \end{pmatrix}.$$

No player has a strictly dominant strategy; therefore there is no strictly dominant equilibrium and the game is not a prisoners' dilemma.

Examples (ctd.)

9. Determine the strictly dominant equilibria for the following game. Is the game a prisoners' dilemma?

$$\begin{pmatrix} -1;-1 & 2;0 \\ 0;2 & 3;3 \end{pmatrix}.$$

Examples (ctd.)

9. Determine the strictly dominant equilibria for the following game. Is the game a prisoners' dilemma?

$$\begin{pmatrix} -1;-1 & 2;0 \\ 0;2 & 3;3 \end{pmatrix}.$$

Both players have a strictly dominant strategy: their second one. So $(2,2)$ is a strictly dominant equilibrium. As $(2,2)$ is weakly Pareto efficient, the game is not a prisoners' dilemma game.

Examples (ctd.)

10. Determine the strictly dominant strategies, the strictly dom. equilibria, the Nash eq. the weakly Pareto eff. strat. profiles and the fully coop. strat. prof. for

$$\begin{pmatrix} -1;0 & -1;1 & 0;0 \\ 2;-2 & -3;3 & -1;3 \\ 4;-3 & 5;-5 & 1;-7 \\ 3;-3 & 3;-5 & -6;8 \end{pmatrix}.$$

Examples (ctd.)

10. Determine the strictly dominant strategies, the strictly dom. equilibria, the Nash eq. the weakly Pareto eff. strat. profiles and the fully coop. strat. prof. for

$$\begin{pmatrix} -1;0 & -1;1 & 0;0 \\ 2;-2 & -3;3 & -1;3 \\ 4;-3 & 5;-5 & 1;-7 \\ 3;-3 & 3;-5 & -6;8 \end{pmatrix}.$$

Strictly dom. strategies for player 1: strategy 3. Strictly dom. strategies for player 1: none. Strictly dom. equilibria: none. Nash eq. strat. profile $(3,1)$. Weakly Pareto eff. strat. prof.: $(1,1),(1,2),(1,3),(2,1)$, $(2,2),(2,3),(3,1),(3,2),(4,1),(4,3)$.

Examples (ctd.)

Fully cooperative strat. prof.: $(2, 3), (4, 3)$.

For example: $(4, 2)$ is not weakly Pareto efficient as $(3, 1)$ is an unanimous Pareto improvement of $(4, 2)$.

11. Determine the weakly Pareto efficient strategy profiles for

$$\begin{pmatrix} 3;8 & 4;8 & 2;3 \\ 1;7 & 2;6 & 8;1 \\ 3;4 & 4;4 & 2;2 \\ 1;1 & 1;-1 & 1;-1 \end{pmatrix}.$$

11. Determine the weakly Pareto efficient strategy profiles for

$$\begin{pmatrix} 3;8 & 4;8 & 2;3 \\ 1;7 & 2;6 & 8;1 \\ 3;4 & 4;4 & 2;2 \\ 1;1 & 1;-1 & 1;-1 \end{pmatrix}.$$

Weakly: (1,1), (1,2), (2,3), (3,2).

Some simple relations

Here are some simple relations between the fundamental notions.

- A player can have at most one strictly dominant strategy, implying that a game can have at most one strictly dominant equilibrium.
- A fully cooperative strategy profile is weakly Pareto efficient.

It should be clear that the relation in the first bullet hold.

Question: are You able to prove that the relation the second second holds?

Some simple relations (ctd.)

Answer:

Some simple relations (ctd.)

Answer: suppose $\mathbf{x}$ is a fully cooperative strategy profile. This means that the total payoff in $\mathbf{x}$ is maximal. This implies that there does not exist another strategy profile with for both players a greater payoff, i.e. there does not exist an unanimous Pareto improvement of $\mathbf{x}$. Thus $\mathbf{x}$ is weakly Pareto efficient.

Please, check in the above examples that these relations indeed hold true.