Urban Economics and Simulations

Slides B

P. v. Mouche

Wageningen University

Spring 2025

Outline

What is game theory?

One may say: game theory is the study of interactive decision making–that is, in situations where each person's action affects the outcome for the whole group.
More precisely:

> *Traditional game theory deals with mathematical models of conflict and cooperation in the real world between at least two rational intelligent players.*

- "Traditional" because of rationality assumption.
- Player: individuals, organisations, countries, animals, computers, ...
- Situations with only one player are studied in classical optimisation theory.

What is game theory? (ctd.)

- Applications.
  - Parlour games
  - Economics: Nobel prices in 1994 for Nash, Harsanyi and Selten, in 2005 for Aumann and in 2007 for Meyerson and Maskin.
  - Sociology, psychology, antropology, politocology.
  - Military strategy.
  - Biology (evolutionary game theory).
  - Design of computer games and robots.
- Game theory provides a language that is very appropriate for conceptual thinking.
- Many game theoretical concepts can be understood without higher mathematics.
- Aim of game theory is to understand/predict how games will be played.

Game in strategic form

Now we start with the formal theory. The important object for us is the notion of game in strategic form .

In a game in strategic form

- there are n players;
- each player has a set of possible strategies;
- choices are made simultaneously and independently.

If player 1 chooses strategy x_1, player 2 strategy x_2, ... , player n strategy x_n, then this leads for player i to a payoff

$$f_i(x_1, \ldots, x_n).$$

Game in strategic form (ctd.)

In fact many economic games have this form. As an example we mention here the topic of industrial organization. The modern theory of industrial organization heavily relies on game theory; various market forms are considered, like that of Cournot Oligopoly . The Cournot Oligopoly is one of the oldest economic games.

A Cournot Oligopoly concerns firms in a competitive setting. There are various variants. Let us here briefly consider the homogeneous duopoly: “duopoly” concerns the assumption of two firms and ‘homogeneous’ that the firms sell the same article.

Cournot Oligopoly (ctd.)

The model is as follows: the firms, 1 and 2, simultaneously and independently supply an amount of the article to the market and then can sell it for a price depending on the total amount. With x_i the amount for firm i, the total amount is $X = x_1 + x_2$ and the price is $p(X)$. The function p is called price function (or inverse demand function). With c_i the cost function of firm i the profit function of firm i, being revenue minus costs, is

$$\pi_i(x_1, x_2) = p(x_1 + x_2)x_i - c_i(x_i).$$

Bimatrix game

This situation becomes simpler in the case of two players, each with a finite number of strategies. Then the game boils down to a bimatrix game.

So in a bimatrix game

- there are 2 players, player 1 and player 2.
- each player has a finite number of strategies: player 1 chooses a row and player 2 a column.
- choices are made simultaneously and independently.

Bimatrix game (ctd.)

Consider, for example, the bimatrix game

$$\begin{pmatrix} 3;3 & 2;2 \\ 7;-1 & -3;1 \\ 1;2 & 12;-9 \end{pmatrix}.$$

- This is a 3×2-bimatrix game, i.e. it has 3 rows and 2 columns.
- Player 1 chooses a row: row 1, row 2 or row 3; so player 1 has 3 strategies. Player 2 chooses a column: column 1 or column 2; so player 2 has 2 strategies.
- At the strategy profile $(3, 2)$, i.e. row 3 and column 2, player 1 has payoff 12 and player 2 has payoff -9.

Many games can be represented in a natural way as a bimatrix game. For example stone-paper-scissors:

$$\begin{pmatrix} 0;0 & -1;1 & 1;-1 \\ 1;-1 & 0;0 & -1;1 \\ -1;1 & 1;-1 & 0;0 \end{pmatrix}$$

Indeed: first strategy is stone, second paper and third scissors. If players make the same choice, then it is draw: payoffs 0 for both. If players make a different choice, then there is a winner with payoff 1 and a looser with payoff -1.

The above game is an example of zero sum game, i.e. a game where the sum of the payoffs at each cell of the bimatrix is zero.

Fundamental notions

Now we are ready to define the notions we need. These notions are not only very useful for our congestion model, but also in the modelling of various economic problems.

Consider a game in strategic form.

- Strategy of a player: completely elaborated plan of playing.
- Strategy profile : for each player a strategy.
- Nash equilibrium : strategy profile such that no player wants to deviate from it.
- Fully cooperative strategy profile : a strategy profile where the sum of the payoffs is maximal.

Instead of “fully cooperative strategy profile” also “social optimum” is used.

Examples

1.

$$\begin{pmatrix} 2;4 & 1;4 & 4;3 & 3;0 \\ 1;1 & 1;2 & 5;2 & 6;1 \\ 1;2 & 0;5 & 3;4 & 7;3 \\ 0;6 & 0;4 & 3;4 & 1;5 \end{pmatrix}.$$

Nash equilibria: $(1,1), (1,2), (2,2)$ and $(2,3)$.

Social optima: $(3,4)$.

Examples (ctd.)

2. $$\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}$$

Nash equilibria: $(3,3)$.
Social optima: $(1,1),(3,2),(3,3)$.

Examples (ctd.)

3. $\begin{pmatrix} 1;0 & 3;1 & 6;0 \\ 2;1 & 4;1 & 8;1 \end{pmatrix}$.

   Nash equilibria: $(2,1),(2,2),(2,3)$.
   Social optima: $(2,3)$.

Examples (ctd.)

4. $\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;3 & 6;1 & 5;2 \end{pmatrix}$

Nash equilibria: none.
Social optima: $(3,1)$.

Examples (ctd.)

5. $\begin{pmatrix} 1;0 & 1;2 & 0;4 \end{pmatrix}$.

   Nash equilibria: $(1,3)$.
   Social optima: $(1,3)$

You can test further Your understanding of these notions with Exercises 1 and 2 (in the file "Exercises A")

John Nash

- John Nash (1928 – 2015).
- Mathematician. (Economist ?)
- Nobel price for economics in 1994, together with Harsanyi and Selten.
- Abel Price for mathematics in 2015. Just after having received it he was killed in a car crash.
- Got this price for his PhD dissertation (27 pages) in 1950.

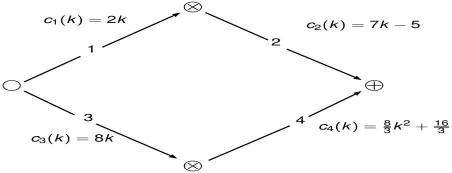
Outline

Introduction

We are going to consider the real world problem of congestion and present a simple game theoretic model for it. In fact, below You can find a quick and efficient route for understanding the very basics of congestion games.

Let us start with a very simple example by considering the following traffic network:

Simple traffic network



Simple traffic network (ctd.)

The intended interpretation is as follows.

- Each morning n commuters want to go from node (i.e. place) $\bigcirc$ to node $\bigoplus$.
- There are 4 roads: 1, 2, 3, 4. The configuration of these roads makes that there are two possible routes for commuting: roads 1–2 (route 1) and roads 3–4 (route 2).
- $c_j(k)$ denotes the costs for a commuter of using road j if k commuters use this road. (So this costs are the same for all commuters who take the road.)

Questions

Questions we want to answer:

- How the commuters will behave?
- Is this behaviour social optimal?

We shall answer these questions by looking to them from a game theoretical perspective. In order to do so we make out of situations as the above one (with only two routes) as follows a game in strategic form.

Of course we assume that the commuters are rational and intelligent. But also that they simultaneously and independently choose a route. Rationality here concerns that commuters want to minimise costs.

Game structure

The commuters are the players and the strategy set of a commuter is the set of routes he can use. So a strategy just is a route. Note that in the above simple model each commuter has the same strategy set. We label (in some way) the commuters by $1, 2, \ldots, n$.

Denote by $(x_1, \ldots, x_n)$ a strategy profile, i.e. commuter 1 chooses x_1, commuter 2 chooses x_2, ... commuter n chooses x_n.

Analysis

First we further suppose $n = 2$, i.e. there are 2 commuters. Denote by $C_1(x_1, x_2)$ the total costs of commuter 1 if this commuter chooses route x_1 and commuter 2 route x_2. Define $C_2(x_1, x_2)$ in the same way.

For example: at the strategy profile, i.e. route profile, $(2, 1)$ (i.e. player 1 takes route 2 and player 2 takes route 1), player 1 has costs $8 \cdot 1$ for road 3 and $\frac{8}{3}1^2 + \frac{16}{3} = 8$ for road 4. Thus $C_1(2, 1) = 8 + 8 = 16$. And for player 2 this leads to $C_2(2, 1) = 2 + 2 = 4$.

Analysis (ctd)

We find

$C_1(1,1) = 13,\ C_2(1,1) = 13$
$C_1(1,2) = 4,\ C_2(1,2) = 16$
$C_1(2,1) = 16,\ C_2(2,1) = 4$
$C_1(2,2) = 32,\ C_2(2,2) = 32$

This can be represented as follows by means of the bimatrix:

$$\begin{pmatrix} 13; 13 & 4; 16 \\ 16; 4 & 32; 32 \end{pmatrix}.$$

Analysis (ctd)

$$\begin{pmatrix} 13;13 & 4;16 \\ 16;4 & 32;32 \end{pmatrix}.$$

A simple game theoretic analysis shows the following.

Prediction of behaviour : both choose route 1.

Social optimal : each commuter chooses a different route.

We see: equilibrium is not social optimal; this is a typical result.

The case of more than two commuters is more difficult to handle. and will

Braess' Paradox

The Braess' Paradox is named after the mathematician Dietrich Braess. It states that adding (removing) a link to a transportation network can increase (decrease) the travel cost for all commuters in the network. It is a counterintuitive phenomenon.

The paradox occurs only in networks in which the commuters operate independently and non cooperatively, in a decentralized manner.

In fact the Braess' Paradox is not limited to traffic flow. It also occurs in other types of 'networks'. In fact it is widespread occurring for example with biological or electricity systems. This makes this paradox extra interesting!

Example from sport: removing a key player from a basketball team can result in the improvement of the team's offensive efficiency. ("When less is actually more.")

Braess’ Paradox (ctd.)

The Braess’ paradox has been observed in various cities, for example in Seoul, New York and Stuttgart.

In New York the often congested 42nd was closed for a parade. People suspected that the closing of this road would lead to the worst traffic jams in history. Instead, the traffic flow actually improved that day.

Braess' Paradox (ctd.)

Let us finally look to the following Youtube video:
`https://www.youtube.com/watch?v=cALezV_Fwi0`

Hotelling Game

Here we consider another example of a model that can deal deal with location: a game theoretical one. It is a discrete variant of the original so-called Hotelling Game.

The (Discrete) Hotelling Game is a game among $n \geq 2$ players that depends on a parameter m, being a positive integer

Consider the $m+1$ points of $H := \{0, 1, \ldots, m\}$ on the real line, to be referred to as *vertices*.

0 1 2 3 4 5 $\cdots$ m

Rule of the game when $n = 2$: 2 players simultaneously and independently choose a vertex. If player 1 (2) chooses vertex x_1 (x_2), then the payoff $f_i(x_1, x_2)$ of player i is the number of vertices that is the closest to his choice x_i; however, a shared vertex, i.e. one that has equal distance to both players, contributes only $1/2$.

Hotelling Game (ctd.)

Various (economic) interpretations of this game are possible. One is the location/vendor/consumer interpretation:

Imagine a stretch of beach on which two ice cream vendors want to sell ice cream. The flavours they offer and the prices they charge are the same, so consumers go to the closest cart. The question for the two vendors is, where should they set up their carts to get the most consumers? In fact there are various variants of this model. Above it is assumed that there is a finite number (i.e. $m+1$) locations where the consumers can locate.

Hotelling Game (ctd).)

Example $m = 7$.

Strategy profile (5,2) :

Payoffs:

$1 + 1 + 1 + 1 = 4$

$1 + 1 + 1 + 1 = 4$

Strategy profile (0,3) :

Payoffs

$1 + 1 = 2$

$1 + 1 + 1 + 1 + 1 + 1 = 6$

Hotelling Game (ctd.)

Example $m = 7$.

Strategy profile (2,6) :

Payoffs:

$1 + 1 + 1 + 1 + \frac{1}{2} = 4\frac{1}{2}$

$\frac{1}{2} + 1 + 1 + 1 = 3\frac{1}{2}$

Strategy profile (3,3) :

Payoffs:

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4$

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4$

Hotelling Game (ctd.)

General rule of the game: n players simultaneously and independently choose a vertex. If player $i = 1, 2, \ldots, n$ chooses vertex x_i, then his payoff $f_i(x_1, x_2, \ldots, x_n)$ is the number of vertices that is the closest to his choice x_i. However, a shared vertex, i.e. one that has the same distance to other players, say k, contributes only $1/(k+1)$.