# 1 Decision making

### Three major elements:

Who is in charge to make the decision? The decision maker (DM):

- one or
- more

What choices the DM has? Alternatives:

- finitely many (discrete problem), $A_1, A_2, \dots, A_m$
- described by continuous variables (continuous problem), like

$$X = \left\{ \underline{x} \mid \underline{x} \in \mathbb{R}^m, \underline{g}(\underline{x}) \leq \underline{0} \right\}$$

What are the consequences of the decision? Objective functions, $\varphi_1, \varphi_2, \dots, \varphi_n$.

### Many cases:

**1 DM with 1 objective:** single objective optimization

**1 DM with multiple objectives:** multiobjective optimization

**multiple DMs with 1 objective each:** game

**multiple DMs with multiple objectives each:** Pareto game

### Games:

DMs are called the **players**;
decision alternatives are called the **strategies**;
objective functions are called the **payoff functions**.

### History:

- John von Neumann (1928)
- John Nash (1950-53)
- Nobel laureates Nash, Selten, Harsanyi (1996)

# 2 Examples of games

#### 1. Prisoners' dilemma

**Players:** Two prisoners who robbed a jewellery store for hire, got arrested, but police does not have enough evidence to convict them with the full crime, only for a much lesser crime of driving a stolen car

**Strategies:** Confess to the police (C) or not (NC)

**Payoffs:**

- If only one confesses, then he recieves very light sentence (1 year), the other gets very harsh sentence (10 years)
- If both confess, they get medium long (5 year) sentence
- If none of them confesses, then they are convicted with a smaller crime, 2 years sentence for each

| 1\ 2 | NC | C |
|---|---|---|
| NC | (-2, -2) | (-10,-1) |
| C | (-1,-10) | (-5,-5) |

$(\varphi_1, \varphi_2)$ in table

**Question:** What to do? Payoff of each depends on the choice of the other player.

**Best response:** Best choice of each player as a function of the choice of the other:

$$R_1 = \begin{cases} \text{C} & \text{if player 2 selects NC} \\ \text{C} & \text{if player 2 selects C} \end{cases}$$

So player 1 always should confess. Same for player 2, who also should always confess.

$\Rightarrow$ C = dominant strategy

$\Rightarrow$ Solution is (C,C) = both confess

However, if the players cooperate, then they can choose (NC, NC) with better payoff for both.

2. **Chicken game**

**Players:** Two kids with motorcycle driving toward each other in a narrow alley

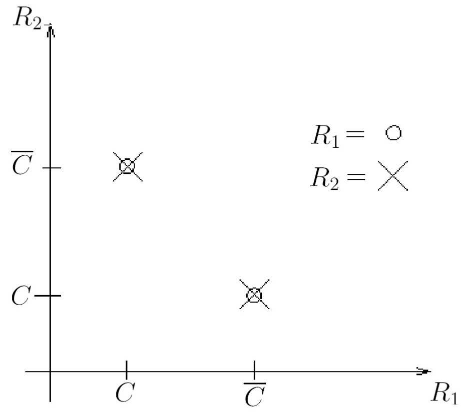
**Strategies:** Give way to the other ($C$=chicken) or not ($\overline{C}$)

**Payoffs:** Being a chicken looks bad in the gang, but by having a crash both might die, even worse outcome

| 1\ 2 | $C$ | $\overline{C}$ |
|---|---|---|
| $C$ | (3, 3) | (2,4) |
| $\overline{C}$ | (4,2) | (1,1) |

$(\varphi_1, \varphi_2)$ in table

$$R_1(C) = \overline{C} \quad R_1(\overline{C}) = C$$
$$R_2(C) = \overline{C} \quad R_2(\overline{C}) = C$$




Both points $(C, \overline{C})$ and $(\overline{C}, C)$ are common in the two best responses, so they called the **equilibria**. No player wants to move away from his equilibrium strategy assuming that the other player keeps his corresponding equilibrium strategy.

Problem & difficulty: in a particular game which equilibrium is selected?

**3. Hunting game**

In a forest, there are rabbits and deers, and two hunters.

**Players:** Two hunters

**Strategies:** Which animal they want to hunt down (D or R)

**Payoffs:**

| 1\ 2 | D | R |
|---|---|---|
| D | (2, 2) | (-1, 1) |
| R | (1,-1) | (0, 0) |

(D, D) = cooperating to get the deer, lot of meat

(D, R) = shooting the rabbit the deer runs away, then player 1 has no success

(R, D) = same but player 2 gets nothing

(R, R) = they get the rabbit

Two equilibria: (D, D) and (R, R)= cooperation

**4. Game of privilege**

Two families take care of a house, e.g. cleaning the common areas (basement, stair-house, etc.)

**Players:** Two families

**Strategies:** Contribute or not in taking care of the house ($C$ or $\overline{C}$)

**Payoffs:**

| $1\backslash 2$ | $C$ | $\overline{C}$ |
|---|---|---|
| $C$ | (3, 3) | (1, 2) |
| $\overline{C}$ | (2, 1) | (0, 0) |

In general:

| $1\backslash 2$ | $C$ | $\overline{C}$ |
|---|---|---|
| $C$ | $(b_2 - c_2, b_2 - c_2)$ | $(b_1 - c_1, b_1)$ |
| $\overline{C}$ | $(b_1, b_1 - c_1)$ | (0, 0) |

Here, $b_1$ = utility if only one contributes

$b_2$ = utility if both contribute

$c_1$ = cost if only one contributes

$c_2$ = cost for one if both contribute

Same as contributing to public goods

In numerical example, equilibrium: (C, C) which is unique

**5. Penalty in soccer**

**Players:** Goalkeeper and the player kicking the penalty (G, K)

**Strategies:** Goalkeeper: moves to left or right (L, R)

Kicker: kicks the ball to the left or right (L, R)

**Payoffs:** Probability of saving for G

| G\ K | L | R |
|---|---|---|
| L | 55 | 20 |
| R | 10 | 80 |

$\varphi_1$

$\varphi_2$=100-$\varphi_1$ $\Rightarrow$ game is equivalent to a zero-sum game. No equilibrium exists

**6. Penalty kick 2**

**Players:** Goalkeeper and player kicking the penalty

**Strategies:** For both: left(L), right(R), middle(M)

**Payoffs:** If goalkeeper's strategy is same as that of the kicker, then he saves, otherwise kicker scores. Zero-sum game with payoff for player 1:

| G\ K | L | R | M |
|---|---|---|---|
| L | 1 | -1 | -1 |
| R | -1 | 1 | -1 |
| M | -1 | -1 | 1 |

$\varphi_1$

No equlibrium exists.

**7. Battle of sexes**

**Players:** Husband and wife, H prefers football game (F), W prefers a movie (M) for an evening. In the morning they leave home without decision and plan to call each other in the afternoon to decide. They could not call each other, so in the evening each of them goes to F or to M.

**Strategies:** Selecting F or M

**Payoffs:**

| H\ W | F | M |
|---|---|---|
| F | (2, 1) | (0, 0) |
| M | (0, 0) | (1, 2) |

Two equilibria: (F, F) and (M, M).

Another interpretation:

**Players:** Two firms with one product each, which complement each other

**Strategies:** Joining the patent of the competitor in order to have compatible products, or not

**Payoffs:** Similar, in the long run it is an advantage to accept the patent of the other firm, since one sale generates the other, however it needs extra work in the short run.

**8. Competition of gas stations**

**Players:** Two gas stations next to each other

**Strategies:** Selling gas with low (L) or high (H) price

**Payoffs:**

| 1\ 2 | H | L |
|---|---|---|
| H | (40, 40) | (10, 50) |
| L | (50, 10) | (20, 20) |

(H, H): both select high price, both get high profit

(H, L) or (L, H): the one with low price gets most of the customers, the other almost nobody

(L, L): they equally share customers with low prices, so both get low profit

Best responses:

$$R_1 = \begin{cases} L & \text{if player 2 selects } H \\ L & \text{if player 2 selects } L \end{cases}$$

$\Rightarrow$ L = dominant strategy for player 1

Similiar, L = dominant strategy for player 2

$\Rightarrow$ The unique equlibrium is (L, L) (same as prisoners' dilemma)

However if they form a coalition and charge high prices, profits increase for both players. This is an illegal act, they violate antitrust regulations.

### 9. Checking tax returns

**Players:** Internal Revenue Service (IRS) and a taxpayer (T)

**Strategies:** For IRS: checking the tax return of the taxpayer or not ($C$ or $\overline{C}$)

For T: cheating or not ($C$ or $\overline{C}$)

**Payoffs:** T should pay 5 thousand \$ as tax. By cheating and being checked, his penalty is also 5 thousand \$; if not checked, then there is no penalty.

For IRS, checking T costs 1 thousand \$. Payoffs:

| IRS\ T | $C$ | $\overline{C}$ |
|---|---|---|
| $C$ | (9, -10) | (4, -5) |
| $\overline{C}$ | (0, 0) | (5, -5) |

No equlibrium exists.

### 10. Checking a worker

**Players:** Supervisor (S) and a worker (W)

**Strategies:** Supervisor: checks the worker or not (C or $\overline{C}$)

Worker: does his job well or not ($J$ or $\overline{J}$)

**Payoffs:** Checking costs amount c to supervisor. Payoffs:

| S\ W | $J$ | $\overline{J}$ |
|---|---|---|
| $C$ | $(8-c, 6)$ | $(6-c, 2)$ |
| $\overline{C}$ | (8, 6) | (6, 8) |

Equilibrium:

(C, J) no, since $8 - c < 8$

($\overline{C}$, J) no, since 6<8

(C, $\overline{J}$) no, since 2<6

($\overline{C}$,$\overline{J}$) yes, since $6 > 6 - c$ and 8>6

**11. Driver and police**

**Players:** Driver and a police officer

**Strategies:** Driver: speeding or not(S or $\overline{S}$)

Police: giving ticket with penalty $p$ or not (T or $\overline{T}$)

**Payoffs:**

| D\ P | $T$ | $\overline{T}$ |
|---|---|---|
| $S$ | $(-p$, 2) | (5, -1)(bad feeling for not catching a speeder) |
| $\overline{S}$ | (0, 0) | (0, 1)(no effort in checking drivers) |

No equlibrium exists

**12. Good citizen**

**Players:** Two people witnessing a serious crime on the street

**Strategies:** Calling the police or not ($C$ or $\overline{C}$)

**Payoffs:**

| 1\ 2 | $C$ | $\overline{C}$ |
|---|---|---|
| $C$ | (7, 7) | (7, 10) |
| $\overline{C}$ | (10, 7) | (0, 0) |

Value of letting the police come to arrest the criminal is 10, but by making the call it decreases by 3 (cost of call + possible revenge of criminal group).

Two equilibria: ($\overline{C}$, $C$) and ($C$, $\overline{C}$) meaning one makes the call and the other walks away. Public goods models are extensions of this one, some of them include mechanisms to avoid free ride (=giving nothing and benefiting)

**13. Waste management**

A firm wants to place dangerous waste on the border between two counties, which would result in $D_1$ and $D_2$ damages to the counties, respectively. The only way to stop it is an intensive lobbying effort by at least one county which would cost $C_1$ or $C_2$.

**Players:** Two counties

**Strategies:** Lobbying ($L$) or not ($\overline{L}$)

**Payoffs:**

| 1\ 2 | $L$ | $\overline{L}$ |
|---|---|---|
| $L$ | $(-C_1, -C_2)$ | $(-C_1, 0)$ |
| $\overline{L}$ | $(0, -C_2)$ | $(-D_1, -D_2)$ |

($L$, $L$) is never equlibrium

($\overline{L}$, $L$) is equlibrium if $-C_2 \geq -D_2 \Leftrightarrow C_2 \leq D_2$

($L$, $\overline{L}$) is equlibrium if $-C_1 \geq -D_1 \Leftrightarrow C_1 \leq D_1$

($\overline{L}$, $\overline{L}$) is equlibrium if $-D_1 \geq -C_1$ and $-D_2 \geq -C_2 \Leftrightarrow C_1 \geq D_1$ and $C_2 \geq D_2$

### 14. Matching pennies

**Players:** Two participants

**Strategy:** Each has a coin, and can show head (H) or tail (T)

**Payoff:** If the coins have identical sides, then player 1 wins \$1, otherwise player 2 wins \$1 from other player:

| 1\2 | H | T |
|---|---|---|
| H | 1 | -1 |
| T | -1 | 1 |

$$\varphi_1$$

$$\varphi_2 = -\varphi_1$$

No equilibrium exists.

### 15. Cooperation in a job

**Players:** Two workers

**Strategies:** Working ($W$) or not($\overline{W}$)

**Payoffs:** Cost of effort (if $W$)=$c \in (0, 1)$

Unit payment is given to both if job is done, and job can be done only if both work

| 1\ 2 | $W$ | $\overline{W}$ |
|---|---|---|
| $W$ | $(1-c,\ 1-c)$ | $(-c,\ 0)$ |
| $\overline{W}$ | $(0,\ -c)$ | $(0,\ 0)$ |

Equilibria: $(W, W)$ and $(\overline{W}, \overline{W})$

### 16. Chain store

**Players:** Chain store (C) & enterpreneur (E).

**Strategies:**

For C: soft or hard on E (S,H)

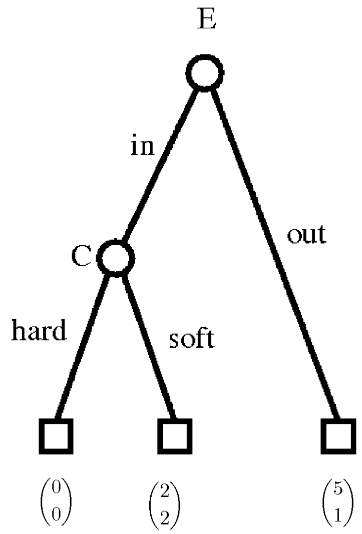
E: stay in business or out (I,O).

**Payoffs:**

| C\ E | I | O |
|---|---|---|
| S | (2, 2) | (5,1) |
| H | (0,0) | (5,1) |

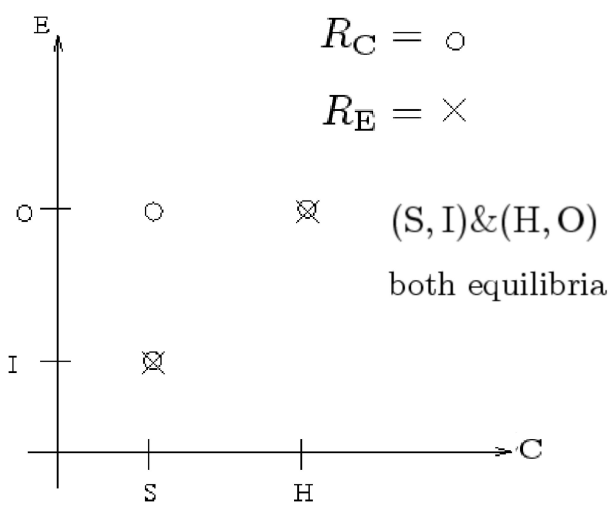
**Normal form** $(n; S_1, \ldots, S_n, \varphi_1, \ldots, \varphi_n)$ giving

- number of players
- strategy sets
- payoff functions.




**Extensive form**, shows development & dynamism of game

$R_{\mathrm{C}}(\mathrm{I}) = \mathrm{S} \quad R_{\mathrm{C}}(\mathrm{O}) = \{\mathrm{S,H}\}$
$R_{\mathrm{E}}(\mathrm{S}) = \mathrm{I} \quad R_{\mathrm{E}}(\mathrm{H}) = \mathrm{O}$




**17. Two-person, zero-sum, discrete games**

**Two players**

**Strategies:** $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$.

| $1\backslash 2$ | $1$ | $\dots$ | $j$ | $\dots$ | $n$ |
|---|---|---|---|---|---|
| $1$ | $a_{11}$ | $\dots$ | $a_{1j}$ | $\dots$ | $a_{1n}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ |
| $i$ | $a_{i1}$ | $\dots$ | $a_{ij}$ | $\dots$ | $a_{in}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ |
| $m$ | $a_{m1}$ | $\dots$ | $a_{mj}$ | $\dots$ | $a_{mn}$ |

$\varphi_1$

**Payoffs:** $\varphi_1(i,j) = a_{ij}$ and $\varphi_2(i,j) = -a_{ij}$

$(a_{ij})$ is equilibrium $\Leftrightarrow$

$a_{ij}$ is the largest among $a_{1j}, \dots, a_{ij}, \dots, a_{mj}$

-$a_{ij}$ is the largest among $-a_{i1}, \dots, -a_{ij}, \dots, -a_{in}$ $\Leftrightarrow$ $a_{ij}$ is smallest among $a_{i1}, \dots, a_{ij}, \dots, a_{in}$

That is, $a_{ij}$ is largest in its column and smallest in its row $\Rightarrow$ **saddle point**
Assume that $a_{ij}$ and $a_{kl}$ are both equilibria. Then

| $1\backslash 2$ | $j$ | $\dots$ | $l$ |
|---|---|---|---|
| $i$ | $a_{ij}$ | $\dots$ | $a_{il}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ |
| $k$ | $a_{kj}$ | $\dots$ | $a_{kl}$ |

$$a_{ij} \geq a_{kj} \geq a_{kl} \text{ and } a_{ij} \leq a_{il} \leq a_{kl}.$$

So, $a_{ij} = a_{kl}$, that is, if multiple equilibria exists, then payoff values are the same.

What is the chance to have equilibrum?

**Theorem 2.1** *Assume that all $a_{ij}$ are independent, identically distributed with a continuous distribution function. Then*

$$\mathbf{P}(\textit{equilibrium exists}) = \mathbf{P}_{mn} = \frac{m!n!}{(m+n-1)!}$$

**Proof.** Notice that

$$\mathbf{P}(\text{all elements } a_{ij} \text{ are different}) = 1$$
$$\mathbf{P}(a_{ij} \text{ is equilibrium}) \text{ is same for all elements}$$
$$\mathbf{P}(\text{equilibrium exists}) = mn\mathbf{P}(a_{11} \text{ is equilibrium})$$

$a_{11}$ is equilibrium if $a_{11}$ is largest in its column and smallest in its row. So if we order the elements of first row and first column in increasing order,

$$a_{m1}, \dots, a_{21}, a_{11}, a_{12}, \dots, a_{1n},$$

then $a_{11}$ must not change position, only the elements before and after $a_{11}$ can be interchanged. So

$$\mathbf{P}(a_{11} \text{ is equilibrium}) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$
$$\mathbf{P}(\text{equilibrium exists}) = mn\frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{m!n!}{(m+n-1)!}$$

■

**Example 2.1**
$m = n = 2$

$$\mathbf{P}_{22} = \frac{2!2!}{3!} = \frac{4}{6} = \frac{2}{3}$$

$m = 2, n = 5$

$$\mathbf{P}_{25} = \frac{2!5!}{6!} = \frac{2 \cdot 120}{720} = \frac{1}{3}$$

$m = 1, n = \textit{arbitrary}$

$$\mathbf{P}_{1n} = \frac{1!n!}{(1+n-1)!} = 1,$$

*the smallest element in the only row is the equilibrium.*
$m = 2, n \geq 2$

$$\mathbf{P}_{2n} = \frac{2!n!}{(2+n-1)!} = \frac{2n!}{(n+1)!} = \frac{2}{n+1} \to 0 \text{ as } n \to \infty$$

*What happens if $m$ increases by 1:*

$$\frac{\mathbf{P}_{m+1,n}}{\mathbf{P}_{m,n}} = \frac{(m+1)!n!}{(m+1+n-1)!} \cdot \frac{(m+n-1)!}{m!n!}$$
$$= \frac{(m+1)!n!(m+n-1)!}{(m+n)!m!n!} = \frac{m+1}{m+n}$$

*which equals 1 if $n = 1$, and is less than 1 if $n \geq 2$ $\Rightarrow$ $\mathbf{P}_{mn} \to 0$ as $m$ or $n$ tends to $\infty$ with other size $\geq 2$.*

▽

**Example 2.2** *With discrete distribution theorem fails:*

$m = n = 2$, $\mathbf{P}(a_{ij} = 0) = p$, $\mathbf{P}(a_{ij} = 1) = 1 - p = q$
*We have $2^4 = 16$ possible matrices:*
*1 is in 0 or 1 position:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

*1 is in 2 positions*

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

*1 is in 3 or 4 positions:*

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

*No equilibrium exists in cases*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

*with probability* $2p^2q^2$, *so*

$$\mathbf{P}(\textit{equilibrium exists}) = 1 - 2p^2q^2$$

$\nabla$

**18.Coin in pocket**

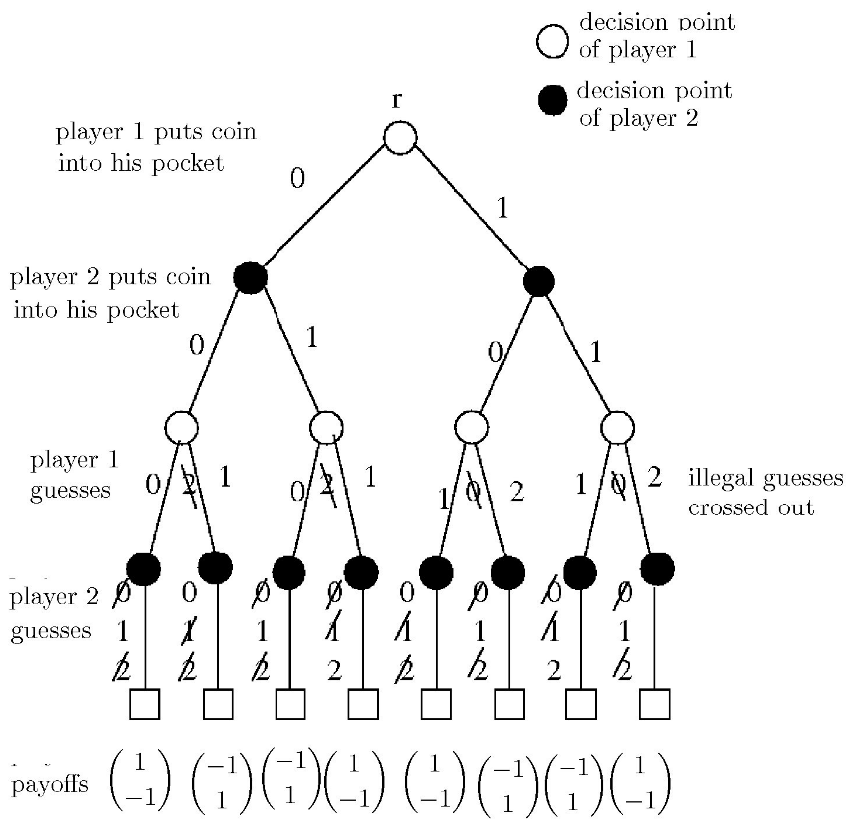
Two players, each puts 0 or 1 coin into his pocket.

**Step 1.** Player 1 guesses total number of coins (no bluffing, so with a coin in his pocket he cannot guess 0).

**Step 2.** Player 2 guesses total number of coins (no bluffing and cannot repeat guess of player 1).

Whoever's guess is correct wins \$1 from other player.

**Extensive form:**



| $1\backslash 2$ | 0 | 1 |
|---|---|---|
| $(0,0)$ | $1$ | $-1$ |
| $(0,1)$ | $-1$ | $1$ |
| $(1,1)$ | $1$ | $-1$ |
| $(1,2)$ | $-1$ | $1$ |

$$\varphi_1$$

**Normal form:**

$$\varphi_2 = -\varphi_1$$

Notice, guess of player 2 was always unique (no bluff, no repeat)

**Strategy of player 1:** $(0,0), (0,1), (1,1), (1,2)$;

**Strategy of player 2:** 0 or 1 (No. of coins in pocket).

No equilibrium exists.

**19. Sharing a pie**

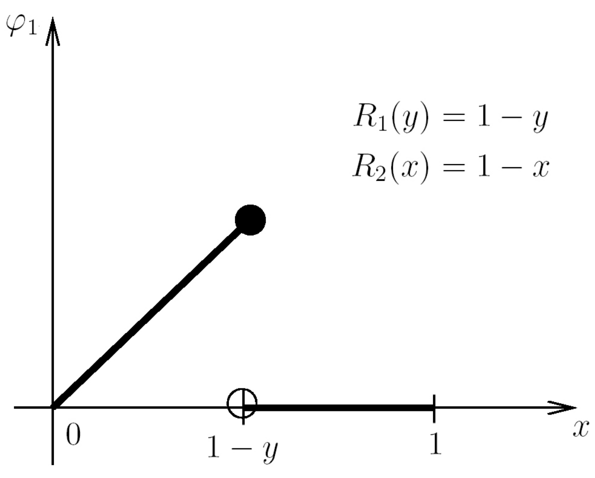
**Players:** 2 people to share a pie of unit size

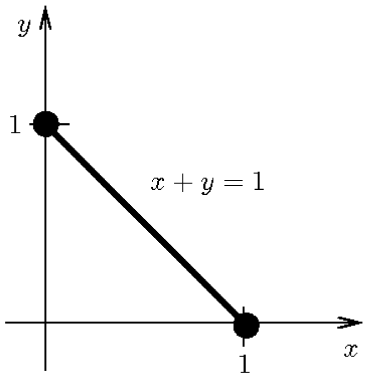
**Strategies:** Requests from the pie, $0 \le x \le 1$, $0 \le y \le 1$

**Payoffs:** If the requests are feasible ($x + y \le 1$) then both get requested amount, if infeasible ($x + y > 1$), then none of them receives anything:

$$\varphi_1(x,y) = \begin{cases} x & \text{if } x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_2(x,y) = \begin{cases} y & \text{if } x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$






Infinitely many equilibria:

$$\{(x,y) \mid 0 \le x, y \le 1, x+y=1\}$$

**20. War game**


**Players:** Airplane (A) and a submarine (S).

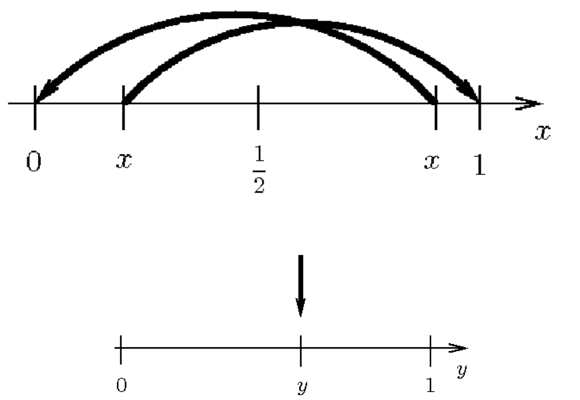
**Strategies:**

For A: $x \in [0,1]$ where to drop a bomb
for S: $y \in [0,1]$ where to hide.

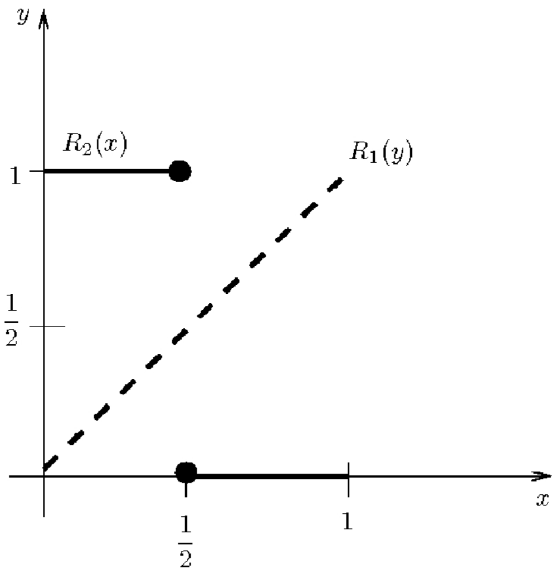
**Payoffs:**

$$\begin{aligned} \varphi_1 &= \alpha\, e^{-\beta(x-y)^2} \quad \text{damage to submarine} \\ \varphi_2 &= -\varphi_1 \end{aligned}$$

**Zero sum game** if $\sum \varphi_k = 0$




$$\begin{aligned} R_1(y) &= \text{drop bomb where submarine is hiding, so} \\ R_1(y) &= y \\ R_2(x) &= \text{hide as far as possible from } x\text{, so} \\ R_2(x) &= \begin{cases} 1 & \text{if } x < \frac{1}{2} \\ 0 & \text{if } x > \frac{1}{2} \\ \{0,1\} & \text{if } x = \frac{1}{2} \end{cases} \end{aligned}$$




No common point, no equilibrium exists.

**21. Modified war game**

**Players:** Airplane (A) and a submarine (S)

**Strategies:**

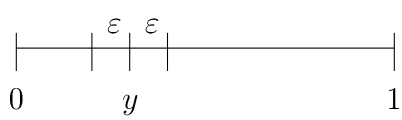
for A: $x \in [0,1]$ where to drop a bomb

for S: $y \in [0,1]$ where to hide

**Payoffs:** if $|x-y| < \varepsilon$, then submarine is destroyed ($\varepsilon > 0$ small):

$$\varphi_1 = \begin{cases} 1 & \text{if } |x-y| < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_2 = -\varphi_1$$




$$R_1(y) = \{x \mid 0 \le x \le 1, y - \varepsilon < x < y + \varepsilon\}$$
$$R_2(x) = \{y \mid 0 \le y \le 1, y \le x - \varepsilon \text{ or } y \ge x + \varepsilon\}$$

No match, no equilibrium.

**22. Selecting a number**

There are $n$ students, each of them has to select a number from the set $\{1, 2, ..., m\}$.

**Players:** $n$ students

**Strategies:** Selecting an integer from set $\{1, 2, ..., m\}$

**Payoff:** If all select the same number, then all get 1$, otherwise none of them gets anything

**Equlibrium:** All selections, except when $n$-1 students select the same number and one selects different number

Proof:

(i) $(i_1, i_2, ..., i_n)$ when $i_1 = i_2 = ... = i_n$ is equilibrium, since if any student changes choice, he will get nothing (as well as all others) $\Rightarrow$ equilibrium

(ii) $(i_1, i_2, ..., i_n)$ when at most $n$-2 students have identical choice, if any one changes strategy, they still will get nothing $\Rightarrow$ equilibrium

(iii) $(i_1, i_2, ..., i_n)$ when $n$-1 students have identical choice, then if the $n$th student changes to the choice of others, then he can increase his payoff $\Rightarrow$ not an equilibrium

**23. Cournot oligopoly**

**Players:** $n$ firms producing same product

**Strategy:** Produced amounts, $x_1, x_2, \ldots, x_n, 0 \leq x_k \leq L_k$

**Payoffs:**

$$\varphi_k(x_1, \ldots, x_n) = x_k p\left(\sum_{l=1}^{n} x_l\right) - C_k(x_k)$$

where

$p$ = price function, decreases in total supply
$C_k$ = cost function of firm $k$.

**Example 2.3** $n = 2$, $C_k(x_k) = x_k + 1$, $0 \leq x_k \leq 5$

$$p(x_1 + x_2) = 10 - (x_1 + x_2)$$

$$\varphi_1 = x_1(10 - x_1 - x_2) - x_1 - 1 = -x_1^2 + 9x_1 - x_1x_2 - 1 \longrightarrow max$$

*Best response of player 1:*

$$\frac{\partial \varphi_1}{\partial x_1} = -2x_1 + 9 - x_2 = 0$$

$$x_1 = \frac{9 - x_2}{2}$$

*always interior, so*

$$R_1(x_2) = \frac{9 - x_2}{2}$$

*Similarly,*

$$R_2(x_1) = \frac{9 - x_1}{2}$$

*Equilibrium:*

$$\begin{array}{lrl} x_1 = \frac{9-x_2}{2} & 2x_1 + x_2 = 9 & x_2 = 9 - 2x_1 \\ x_2 = \frac{9-x_1}{2} & 2x_2 + x_1 = 9 & \\ \hline & 18 - 4x_1 + x_1 = 9 & \\ & 9 = 3x_1 & \\ & x_1 = 3 & x_2 = 3 \end{array}$$

$\bigtriangledown$

### 24. Commercial fishing

**Players:** $n$ firms fishing in an open sea

**Strategies:** Number of boats, $h_k$, sent for fishing by firm $k$

**Payoffs:** Profit per boat $=A - B\sum_{l=1}^{n} h_l$

Cost per boat$=C$ $(A > C)$

$\Rightarrow$ profit of firm $k$: $\varphi_k = h_k \left(A - B\sum_{l=1}^{n} h_l\right) - Ch_k$

Best response:

$\frac{\partial\varphi_k}{\partial h_k} = A - B\sum_{l\neq k} h_l - 2Bh_k - C = 0$ ($k$=1, 2, ..., $n$)

Assuming nonnegative solution this is a system of linear equations. Let $H = \sum_{l=1}^{n} h_l$, then

$A - BH - Bh_k - C = 0$

So, $h_k = -H + \frac{A-C}{B}$

By adding for all $k$, $H = -nH + \frac{n(A-C)}{B}$

$(n+1)H$=$\frac{n(A-C)}{B}$ $\Rightarrow$ $H$=$\frac{n(A-C)}{(n+1)B}$

By symmetry: $h_k = \frac{A-C}{(n+1)B} > 0$

### 25. Single-product oligopolies with product differentiation

**Players:** $n$ firms producing related products

**Strategies:** Produced quatities, $0 \leq x_k \leq L_k$

**Payoffs:** Price functions, $P_k(x_1, ..., x_n)$

Cost functions, $C_k(x_k)$

Profit of firm $k$, $\varphi_k = x_k P_k(x_1, ..., x_n) - C_k(x_k)$

Checking conditions of Nikaido-Isoda theorem (see Chapter 6):

(i) Strategy set of player $k$ is $S_k = [0, L_k]$, which is convex, closed, bounded in one-dimensional space

(ii) $\varphi_k$ is continuous if both $P_k$ and $C_k$ are continuous

(iii) $\varphi_k$ is concave in $x_k$ if

$\frac{\partial^2\varphi_k}{\partial {x_k}^2} = \frac{\partial\left(P_k + x_k\frac{\partial P_k}{\partial x_k} - C'_k\right)}{\partial x_k}$=$2\frac{\partial P_k}{\partial x_k} + x_k\frac{\partial^2 P_k}{\partial {x_k}^2} - C''_k \leq 0$

$\Rightarrow$ under (i), (ii), (iii) there is at least one equilibrium.

### 26. Bertrand oligopoly

**Players:** $n$ firms producing similar products

**Strategies:** Setting prices for own products, $0 \leq p_k \leq P_k$

**Payoffs:**

$$\varphi_k(p_1, p_2, \ldots, p_n) = p_k d_k(p_1, \ldots, p_n) - C_k\left(d_k(p_1, \ldots, p_n)\right)$$

where $d_k$ = demand of the product made by firm $k$.

### 27. Special duopoly

**Players:** 2 firms, price setting.

**Strategies:** Prices but giving discounts to faithful customers.

**Payoffs:**

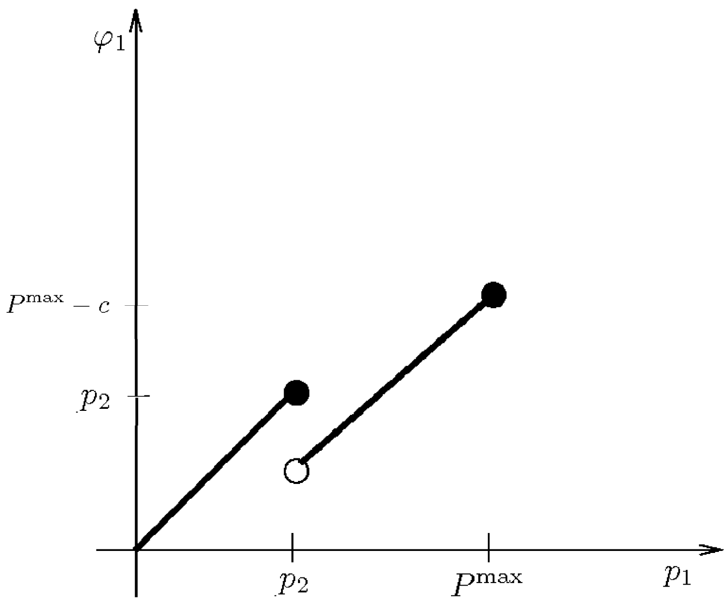
$$\varphi_1 = \begin{cases} p_1 & \text{if } p_1 \le p_2 \\ p_1 - c & \text{if } p_1 > p_2 \end{cases}$$

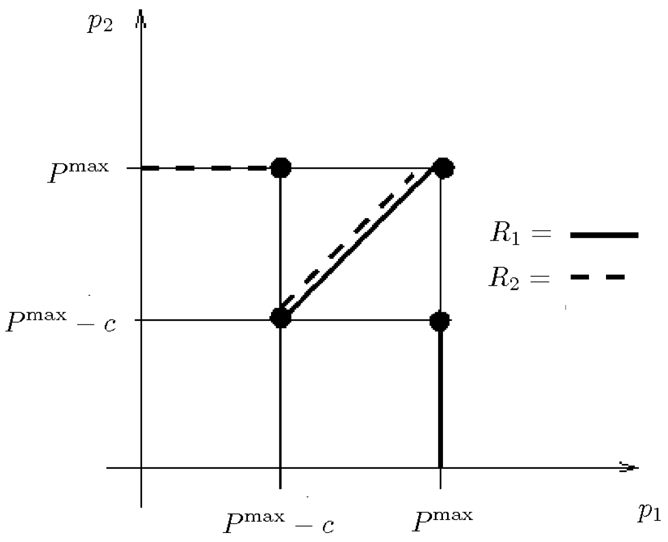
$$\varphi_2 = \begin{cases} p_2 & \text{if } p_2 \le p_1 \\ p_2 - c & \text{if } p_2 > p_1 \end{cases}$$

Assume max price, $P^{\max}$ is large enough:

$$R_1(p_2) = \begin{cases} p_2 & \text{if } p_2 > P^{\max} - c \\ P^{\max} & \text{if } p_2 < P^{\max} - c \\ \{p_2, p^{\max}\} & \text{if } p_2 = P^{\max} - c \end{cases}$$

$R_2$ is the same






Infinitely many equilibria:

$$\{(p_1, p_2) \mid P^{\max} - c \leq p_1 = p_2 \leq P^{\max}\}$$

**28. Price war**

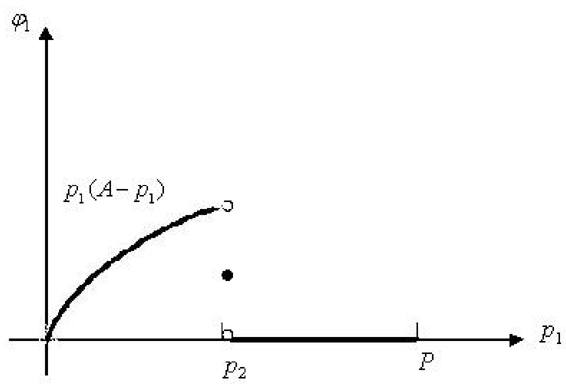
**Players:** 2 firms

**Strategies:** Selected prices $p_1, p_2 \in [0, P]$

**Payoffs:** Demand $D = A - p$ ($p$=min$\{p_1$,$p_2\}$, $A \geq 2P$)

$$\varphi_1 = \begin{cases} p_1(A - p_1) & \text{if } p_1 < p_2 \text{ everybody buys for lower price} \\ \frac{1}{2}p_1(A - p_1) & \text{if } p_1 = p_2 \text{ they share market} \\ 0 & \text{if } p_1 > p_2 \text{ nobody buys for higher price} \end{cases}$$

$\frac{\partial(p_1(A-p_1))}{\partial p_1}$=$A - 2p_1 > 0$, $\frac{\partial^2(p_1(A-p_1))}{\partial {p_1}^2} = -2 < 0$




No best response, no equilibrium exists.

**29. Sharing 100$**

**Players:** Two people

**Strategies:** Step 1. Player 1 gives an offer $x_1 \in \{0, 25, 50, 75, 100\}$ to player 2, simultaneously player 2 gives a minimum acceptable $x_2$ amount

**Payoffs:** If $x_1 < x_2$, they get nothing, and if $x_1 \geq x_2$, then player 2 gets $x_1$ and player 1 keeps $100 - x_1$

| 1\ 2 | 0 | 25 | 50 | 75 | 100 |
|---|---|---|---|---|---|
| 0 | (100, 0) | (0, 0) | (0,0) | (0,0) | (0,0) |
| 25 | (75, 25) | (75, 25) | (0,0) | (0,0) | (0,0) |
| 50 | (50, 50) | (50, 50) | (50, 50) | (0, 0) | (0, 0) |
| 75 | (25, 75) | (25, 75) | (25, 75) | (25, 75) | (0,0) |
| 100 | (0, 100) | (0, 100) | (0,100) | (0,100) | (0,100) |

Equilibria: (0,0), (25,25), (50,50),(75,75), (100,100)

**30. Pick a number**

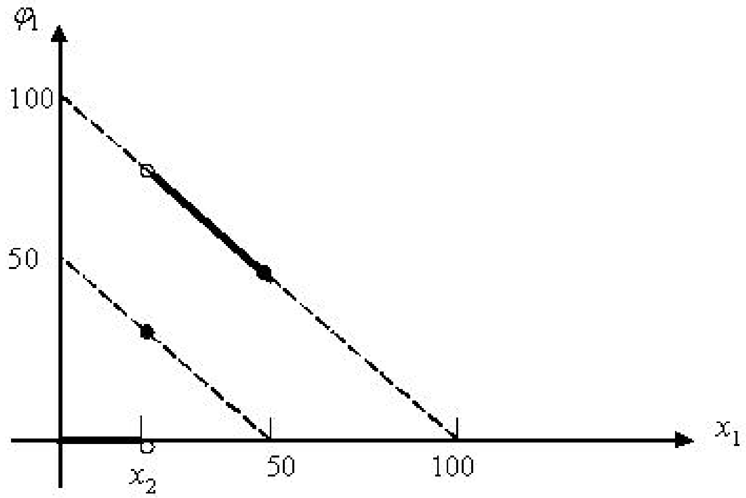
**Players:** Two people

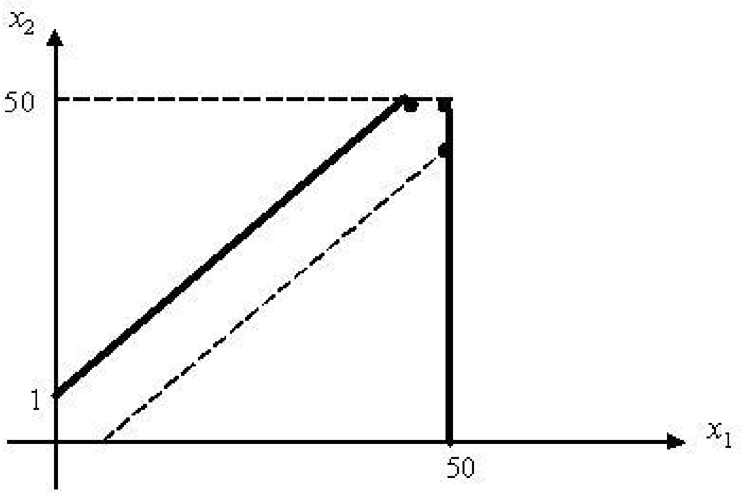
**Strategies:** $x_1, x_2$ positive integers $\leq 50$

**Payoffs:** If $x_1 = x_2$, then both get 50-$x_1$, $\varphi_1 = \varphi_2 = 50 - x_1$

If $x_1 > x_2$, then $\varphi_1 = 100 - x_1$ and $\varphi_2 = 0$

If $x_1 < x_2$, then $\varphi_1 = 0$ and $\varphi_2 = 100 - x_2$






Discrete problem!!!

$$R_1(x_2) = \begin{cases} all & \text{if } x_2 = 50 \\ x_2 + 1 & \text{if } x_2 < 50 \end{cases}$$

Three equilibria: (50,50), (49,50), (50,49)

**31.Quality control** A salesman sells an equipment to a customer with rules:

- If equipment is good, customer pays $\$\alpha$ to salesman
- if equipment is defective, salesman pays $\$\beta$ to customer.

Equipment has 3 parts, they can be defective with equal probabilities.

**Players:** Salesman and equipment (S & E )

**Strategies:**

For S, how many parts to check before selling equipment: 0, 1, 2 or 3. Cost of each checking is $\$\gamma$

For E, how many parts are defective: 0, 1, 2 or 3

**Payoffs:**

$\varphi_1 =$ expected profit of salesman

$\varphi_2 = -\varphi_1$

| $S\backslash E$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\alpha$ | $-\beta$ | $-\beta$ | $-\beta$ |
| 1 | $\alpha-\gamma$ | $-\frac{2}{3}\beta-\gamma$ | $-\frac{1}{3}\beta-\gamma$ | $-\gamma$ |
| 2 | $\alpha-2\gamma$ | $-\frac{1}{3}\beta-\frac{5}{3}\gamma$ | $-\frac{4}{3}\gamma$ | $-\gamma$ |
| 3 | $\alpha-3\gamma$ | $-2\gamma$ | $-\frac{4}{3}\gamma$ | $-\gamma$ |

$A_{11}$**:** defective part is not found with probability $\frac{2}{3}$

$A_{12}$**:** defective part is not found with probability $\frac{1}{3}$

$A_{21}$: defective part is found either in first or second checking, or not

$$\frac{1}{3}\Big(-\gamma\Big)+\frac{2}{3}\left[\frac{1}{2}\Big(-2\gamma\Big)+\frac{1}{2}\Big(-2\gamma-\beta\Big)\right]$$

$A_{22}$: same principle

$$\frac{2}{3}\Big(-\gamma\Big)+\frac{1}{3}\Big(-2\gamma\Big)$$

$A_{31}$: defective part is found either in first, second or third checking

$$\frac{1}{3}\Big(-\gamma\Big)+\frac{2}{3}\left[\frac{1}{2}\Big(-2\gamma\Big)+\frac{1}{2}\Big(-3\gamma\Big)\right]$$

$A_{32}$: defective part is found either in first or second checking

$$\frac{2}{3}\Big(-\gamma\Big)+\frac{1}{3}\Big(-2\gamma\Big)$$

Equilibrium?

**Row 0** has three smallest elements $a_{01}, a_{02}, a_{03}$

$a_{01}$ is equilibrium if

$$\begin{array}{ccc} -\beta \geq -\tfrac{2}{3}\beta-\gamma, & -\beta \geq -\tfrac{1}{3}\beta-\tfrac{5}{3}\gamma, & -\beta \geq -2\gamma \\ \beta \leq 3\gamma & \beta \leq \tfrac{5}{2}\gamma & \boxed{\beta \leq 2\gamma} \end{array}$$

$a_{02}$ is equilibrium if

$$\begin{array}{ccc} -\beta \geq -\tfrac{1}{3}\beta-\gamma, & -\beta \geq -\tfrac{4}{3}\gamma, & -\beta \geq -\tfrac{4}{3}\gamma \\ \beta \leq \tfrac{3}{2}\gamma & \boxed{\beta \leq \tfrac{4}{3}\gamma} & \end{array}$$

$a_{03}$ is equilibrium if

$$-\beta \geq -\gamma, \quad \boxed{\beta \leq \gamma}$$

**Row 1** has one smallest element $a_{11}$

$a_{11}$ is equilibrium if

$$\begin{array}{ccc} -\tfrac{2}{3}\beta-\gamma \geq -\beta, & -\tfrac{2}{3}\beta-\gamma \geq -\tfrac{1}{3}\beta-\tfrac{5}{3}\gamma, & -\tfrac{2}{3}\beta-\gamma \geq -2\gamma \\ \beta \geq 3\gamma & \beta \leq 2\gamma & \beta \leq \tfrac{3}{2}\gamma \end{array}$$

contradiction

**Row 2** has two potential smallest elements $a_{20}, a_{21}$

$a_{20}$ is not equilibrium, it is not largest in column

$a_{21}$ is equilibrium if

$$\begin{array}{ccc} -\tfrac{1}{3}\beta-\tfrac{5}{3}\gamma \geq -\beta, & -\tfrac{1}{3}\beta-\tfrac{5}{3}\gamma \geq -\tfrac{2}{3}\beta-\gamma, & -\tfrac{1}{3}\beta-\tfrac{5}{3}\gamma \geq -2\gamma \\ \beta \geq \tfrac{5}{2}\gamma & \beta \geq 2\gamma & \beta \leq \gamma \end{array}$$

contradiction

and

$$-\tfrac{1}{3}\beta-\tfrac{5}{3}\gamma \leq \alpha-2\gamma$$

irrelevant

**Row 3** has two possible smallest elements $a_{30}$ and $a_{31}$

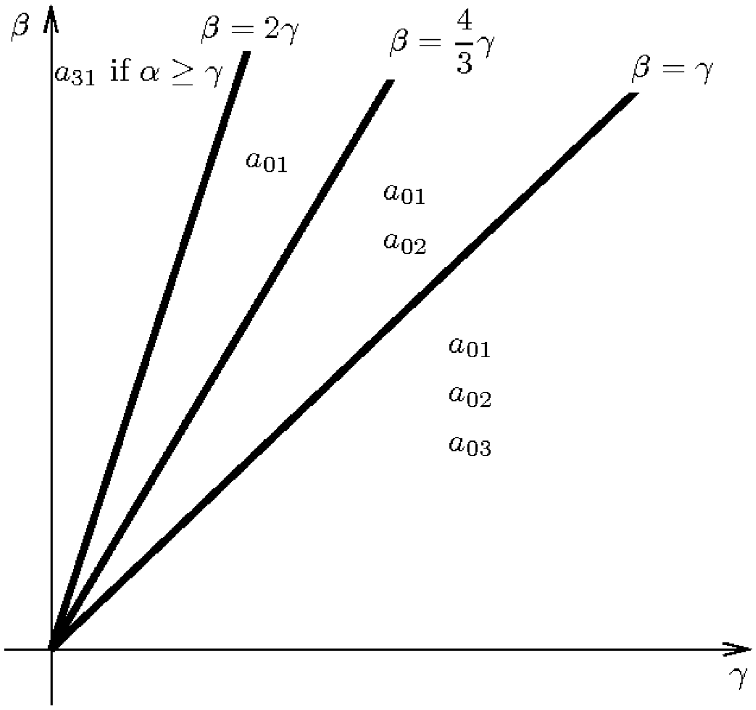
$a_{30}$ is not largest in column, it is not equilibrium

$a_{31}$ is equilibrium if

$$-2\gamma \geq -\beta, \quad -2\gamma \geq -\tfrac{2}{3}\beta - \gamma, \quad -2\gamma \geq -\tfrac{1}{3}\beta - \tfrac{5}{3}\gamma$$
$$\boxed{\beta \geq 2\gamma} \qquad \beta \geq \tfrac{3}{2}\gamma \qquad \beta \geq \gamma$$

and

$$-2\gamma \leq \alpha - 3\gamma$$
$$\boxed{\alpha \geq \gamma}$$




**32.Timimg game**

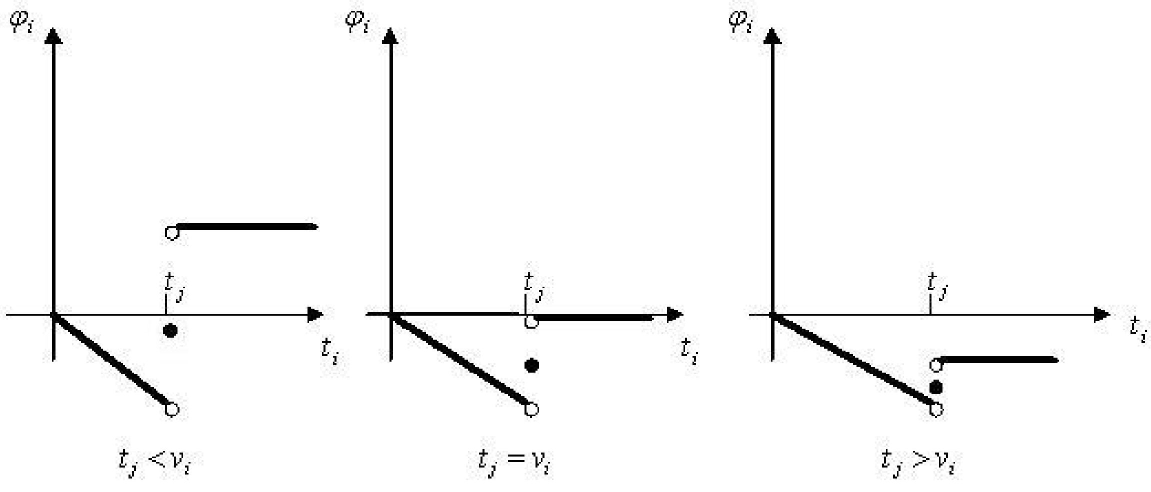
Two players want to get an object valued as $v_1$ and $v_2$ by them ($v_1 \neq v_2$). Both want to wait as long as possible hoping that the other will give up fighting for the object, so he can get it. (Price war, isolating a community in war, etc.)

**Players:** Two agents

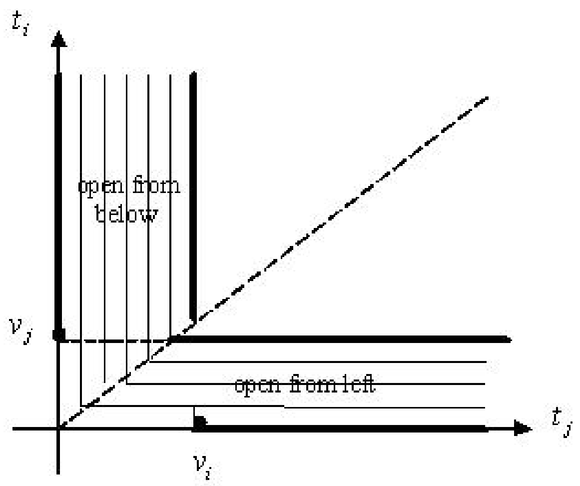
**Strategies:** When to give up, $t_1$ and $t_2 (\geq 0)$

**Payoffs:**

$$\varphi_i = \begin{cases} -t_i & \text{if } t_i < t_j \text{ (he gives up)} \\ \frac{1}{2}v_i - t_i & \text{if } t_i = t_j \text{ (he has } \frac{1}{2} \text{ chance to get the item)} \\ v_i - t_j & \text{if } t_i > t_j \text{(other gives up earlier)} \end{cases}$$




$$R_i(t_j) = \begin{cases} (t_j, \infty) & \text{if } t_j < v_i \\ \{0\} \cup (t_j, \infty) & \text{if } t_j = v_i \\ 0 & \text{if } t_j > v_i \end{cases}$$




Equilibria are:

$\{t_j \in [v_i, \infty)$ and $t_i = 0$ $\} \cup \{t_j = 0$ and $t_i \in [v_j, \infty)\}$

**33.Position game**

Two manufacturers produce one product each with a quality parameter $x_1$ and $x_2$. Customers' expectation is about quality value M. That player wins who's quality is closer to customers' expectation

**Players:** Two manufacurers

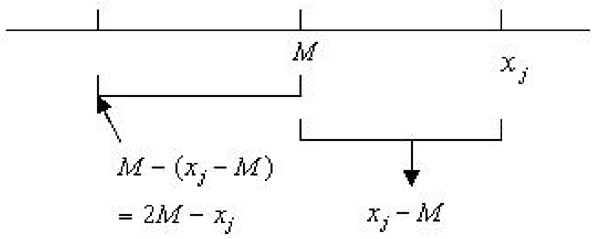
**Strategies:** $x_1$ and $x_2$, quality parameters

**Payoffs:**

$$\varphi_i = \begin{cases} 1 & \text{if } |x_i - M| < |x_j - M| \\ 0 & \text{if } |x_i - M| = |x_j - M| \\ -1 & \text{if } |x_i - M| > |x_j - M| \end{cases}$$

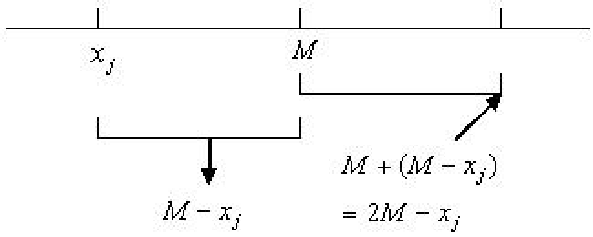
Case 1 occurs if

$x_j > M$:



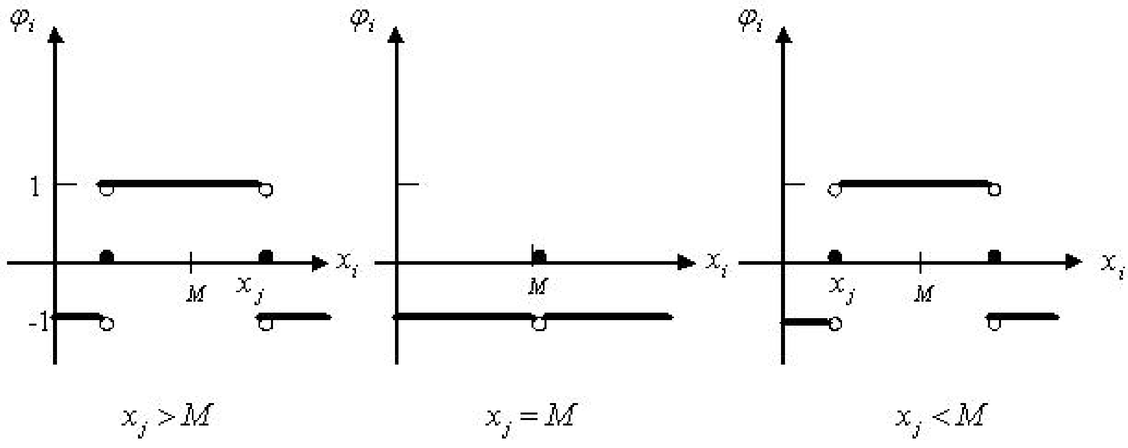

$$2M - x_j < x_i < x_j$$

$x_j < M$:



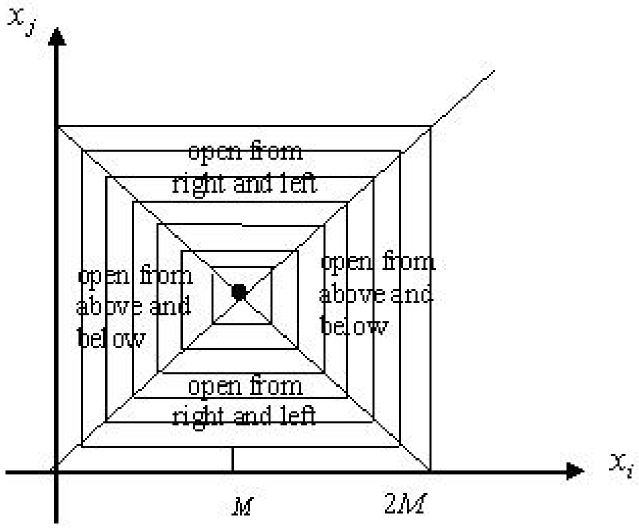

$$x_j < x_i < 2M - x_j$$

$x_j = M$: never occurs




$$R_i(x_j) = \begin{cases} (2M - x_j, x_j) & \text{if } x_j > M \\ M & \text{if } x_j = M \\ (x_j, 2M - x_j) & \text{if } x_j < M \end{cases}$$

(vertically shaded region)




Unique equilibrium:$x_i = x_j = M$

**34.Location game**

Two icecream sellers have to select locations for their shops on interval [0,1]. The potential buyers are uniformly placed on the interval, and there are infinitely many. Each buyer goes to the closer shop to buy icecream.

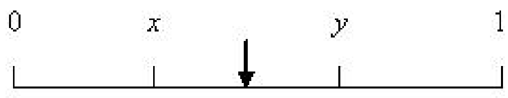
**Players:** Two icecream sellers

**Strategies:** Locations of shops, $x, y \in [0, 1]$

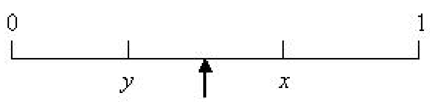
**Payoffs:**

$$\varphi_1 = \begin{cases} \frac{x+y}{2} & \text{if } x < y \\ \frac{1}{2} & \text{if } x = y \\ 1 - \frac{x+y}{2} & \text{if } x > y \end{cases}$$

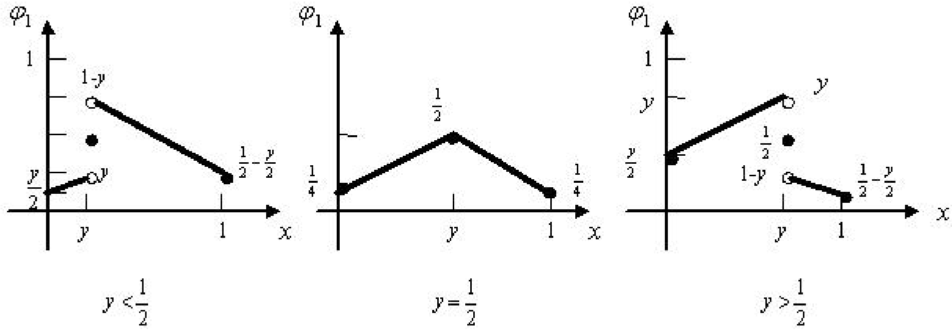
For case 1:




For case 3:




$$\varphi_2 = \begin{cases} 1 - \frac{x+y}{2} & \text{if } x < y \\ \frac{1}{2} & \text{if } x = y \\ \frac{x+y}{2} & \text{if } x > y \end{cases}$$




$R_1(y) = \frac{1}{2}$ only if $y = \frac{1}{2}$

Similiarly, $R_2(x) = \frac{1}{2}$ only if $x = \frac{1}{2}$

Unique equilibrium:$x = y = \frac{1}{2}$

**35.Advertisement**

Two firms compete for $m$ markets with number of potential customers $a_1 > a_2 > \cdots > a_m$.

**Players:** 2 firms

**Strategies:** Which market is selected to conduct intensive advertisement (they can select only one), $1 \leq i, j \leq m$

**Payoffs:** If they advertise in different markets, then they get all customers, and if they advertise on the same market, they share customers:

| $1\backslash 2$ | 1 | 2 | $\dots$ | $m$ |
|---|---|---|---|---|
| 1 | $p_1 a_1$ | $a_1$ | $\dots$ | $a_1$ |
| 2 | $a_2$ | $p_2 a_2$ | $\dots$ | $a_2$ |
| $\vdots$ | | $\vdots$ | | $\vdots$ |
| $m$ | $a_m$ | $a_m$ | $\dots$ | $p_m a_m$ |

$\varphi_1$

| $1\backslash 2$ | 1 | 2 | $\dots$ | $m$ |
|---|---|---|---|---|
| 1 | $q_1 a_1$ | $a_2$ | $\dots$ | $a_m$ |
| 2 | $a_1$ | $q_2 a_2$ | $\dots$ | $a_m$ |
| $\vdots$ | | $\vdots$ | | $\vdots$ |
| $m$ | $a_1$ | $a_2$ | $\dots$ | $q_m a_m$ |

$\varphi_2$

$(i, j)$ is equilibrium if corresponding element in $\varphi_1$ is largest in its column and that in $\varphi_2$ is largest in its row.

In column 1 the largest is

$(1, 1)$ if $p_1 a_1 \geq a_2$

$(2, 1)$ if $a_2 \geq p_1 a_1$.

In columns $2, \dots, m$, elements $(1,2), \dots, (1,m)$ are the largest.

In row 1, the largest is

$(1,1)$ if $q_1 a_1 \geq a_2$

$(1,2)$ if $a_2 \geq q_1 a_1$.

In rows $2, \dots, m$, elements $(2,1), \dots, (m,1)$ are the largest.

Only matches are

$$\begin{aligned} (1,1) &\Leftrightarrow p_1 a_1 \geq a_2 \text{ and } q_1 a_1 \geq a_2 \\ (2,1) &\Leftrightarrow a_2 \geq p_1 a_1 \\ (1,2) &\Leftrightarrow a_2 \geq q_1 a_1. \end{aligned}$$

**Modified game:** $\varphi_1$ is as before but firm 1 believes that firm 2 wants to damage it, so it believes $\varphi_2 = -\varphi_1$

Equilibrium in $\varphi_1$ matrix is largest in its column but smallest in its row:

Smallest elements in rows are $(1,1), (2,2), \dots, (m,m)$

Largest elements in columns are:

in column 1,

$(1,1)$ if $p_1 a_1 \geq a_2$

$(2,1)$ if $a_2 \geq p_1 a_1$

in columns $2, \dots, m$, only $(1,2), \dots, (1,m)$

Only match is $(1,1) \Leftrightarrow p_1 a_1 \geq a_2$ otherwise there is no equilibrium.

### 36. Advertisement budget allocation

**Players:** 2 competing firms on $m$ markets

**Strategies:** Amounts spent on the markets in advertisement

$$(x_1, \dots, x_m) \text{ and } (y_1, \dots, y_m)$$

**Payoffs:**

$$\begin{aligned} \varphi_1 &= \sum_{i=1}^{m} \frac{x_i a_i}{x_i + y_i + z_i} \\ \varphi_2 &= \sum_{j=1}^{m} \frac{y_j a_j}{x_j + y_j + z_j} \end{aligned}$$

where $z_i =$ total spending of others in market $i$.

### 37. Market share

**Players:** Two firms compete for a business of unit value

**Strategies:** Efforts in order to get larger portions of the business (e.g. market) $x, y \geq 0$

**Payoffs:**

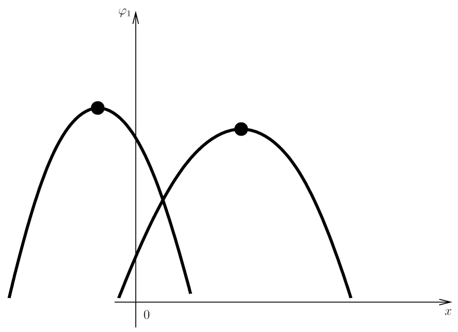
$$\begin{aligned}\varphi_1 &= \frac{x}{x+y} - x \text{ (business share value – cost)}\\ \varphi_2 &= \frac{y}{x+y} - y\end{aligned}$$

**Best responses:**

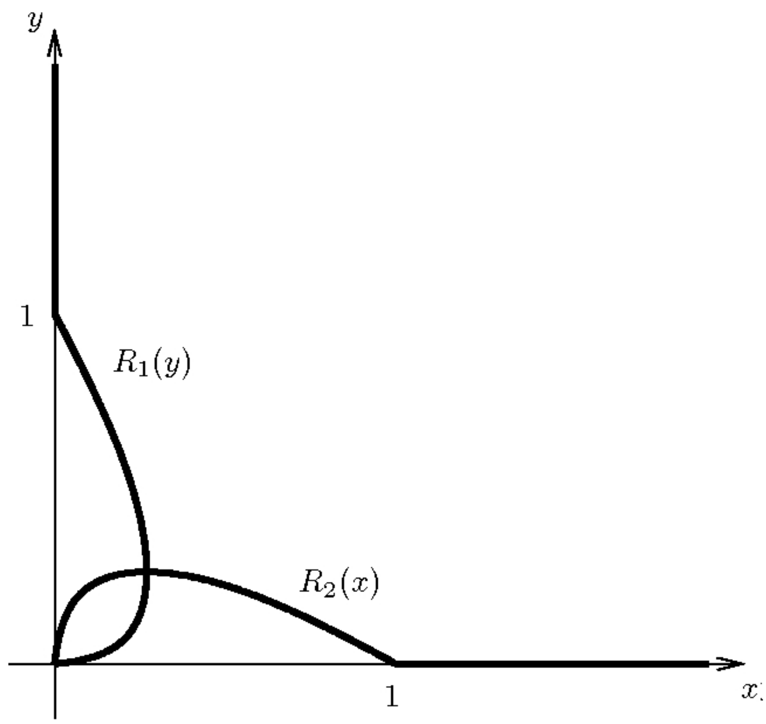
$$\begin{aligned}\frac{\partial \varphi_1}{\partial x} = \frac{1 \cdot (x+y) - x \cdot 1}{(x+y)^2} - 1 &= \frac{y}{(x+y)^2} - 1 = 0\\ (x+y)^2 &= y\\ x+y &= \sqrt{y}\\ x &= \sqrt{y} - y \quad \text{stationary point}\end{aligned}$$

$$\frac{\partial^2 \varphi_1}{\partial x^2} = \frac{-y \cdot 2(x+y)}{(x+y)^4} \quad < 0 \quad \text{as } y > 0.$$

$\varphi_1$ is strictly concave, vertex can be negative or nonnegative:




$$R_1(y) = \begin{cases} 0 & \text{if } y \geq 1 \\ \sqrt{y} - y & \text{if } y \leq 1 \end{cases}, \quad R_2(x) = \begin{cases} 0 & \text{if } x \geq 1 \\ \sqrt{x} - x & \text{if } x \leq 1 \end{cases}$$



Intercepts are $(0,0)$ and $(\frac{1}{4},\frac{1}{4})$

$$\begin{array}{ccc} x=\sqrt{y}-y & x+y=\sqrt{y} & 2x=\sqrt{x} \\ y=\sqrt{x}-x & x+y=\sqrt{x} & 4x^2-x=0 \\ & x=y & x(4x-1)=0 \\ & & x=0 \text{ or } x=\frac{1}{4} \end{array}$$

Equilibrium is $x=y=\frac{1}{4}$.

### 38. Inventory control

**Players:** A retailer and a wholesaler

**Strategies:** Inventories, $y, z \geq 0$

**Payoffs:** Random demand $x$ with pdf $f(x)$

$$\varphi_1 = a_1 \int_0^y xf(x)dx + \int_y^{y+z} \big[a_1 y + a_2(x-y)\big] f(x)dx + \int_{y+z}^{\infty} \big[a_1 y + a_2 z\big] f(x)dx - b_1 y$$

**1st term:** $a_1 =$ unit profit from own inventory $(x \leq y)$

**2nd term:** $a_2 =$ unit profit from back order $(y < x \leq y+z)$

**3rd term:** same $(x > y+z)$

**4th term:** $b_1 =$ unit inventory cost of retailer

$$\varphi_2 = a_3 \int_y^{y+z} (x-y)f(x)dx + \int_{y+z}^{\infty} a_3 z f(x)dx - b_2 z$$

**1st term:** $a_3 =$ unit profit of wholesaler from back order $(y < x \leq y+z)$

**3rd term:** $b_2 =$ unit inventory cost of wholesaler.

### 39. Price strategy

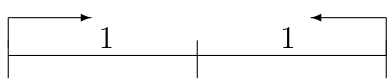
**Players:** $n$ firms producing similar goods

**Strategy:** Time varying price of own product, $p_k(t) \in [0, P_k]$

**Payoffs:** Let $\Psi_k(p_1, \dots, p_n)$ be demand of good $k$, then

$$\varphi_k = \int_0^T \Psi_k(p_1(t), \dots, p_n(t)) p_k(t) dt = \text{total revenue.}$$

### 40. Duel without sound




Two duelists are placed 2 units from each other, each has a gun with 1 bullet in it. For a signal they start walking toward each other, and can shoot at any time. Their speeds are equal, guns have silencers.

**Players:** Two participants

**Strategies:** Where to shoot, $0 \leq x, y \leq 1$

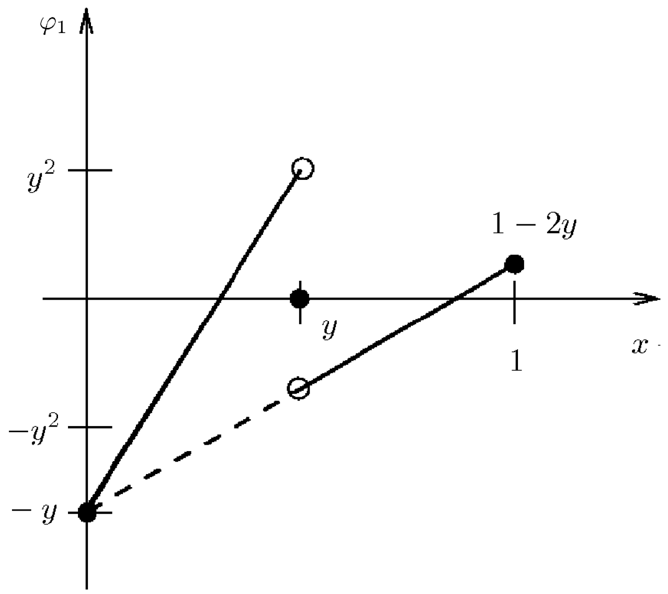
**Payoffs:** Hitting probabilities $P_1(x)$ and $P_2(y)$, then

$$\varphi_1 = \begin{cases} P_1(x)\cdot 1-(1-P_1(x))\cdot P_2(y) & \text{if } x<y \\ P_1(x)-P_2(y) & \text{if } x=y \\ P_2(y)\cdot(-1)+(1-P_2(y))P_1(x) & \text{if } x>y \end{cases}$$

$$\varphi_2 = \begin{cases} P_2(y)\cdot 1-(1-P_2(y))\cdot P_1(x) & \text{if } y<x \\ P_2(y)-P_1(x) & \text{if } y=x \\ -P_1(x)+(1-P_1(x))P_2(y) & \text{if } y>x \end{cases}$$

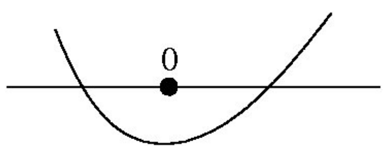
**Example 2.4** $P_1(x)=x, P_2(y)=y$

$$\varphi_1 = \begin{cases} x\cdot 1-(1-x)\cdot y = x-y+xy & \textit{if } x<y \\ x-y=x-y & \textit{if } x=y \\ -y+(1-y)x = x-y-xy & \textit{if } x>y \end{cases}$$

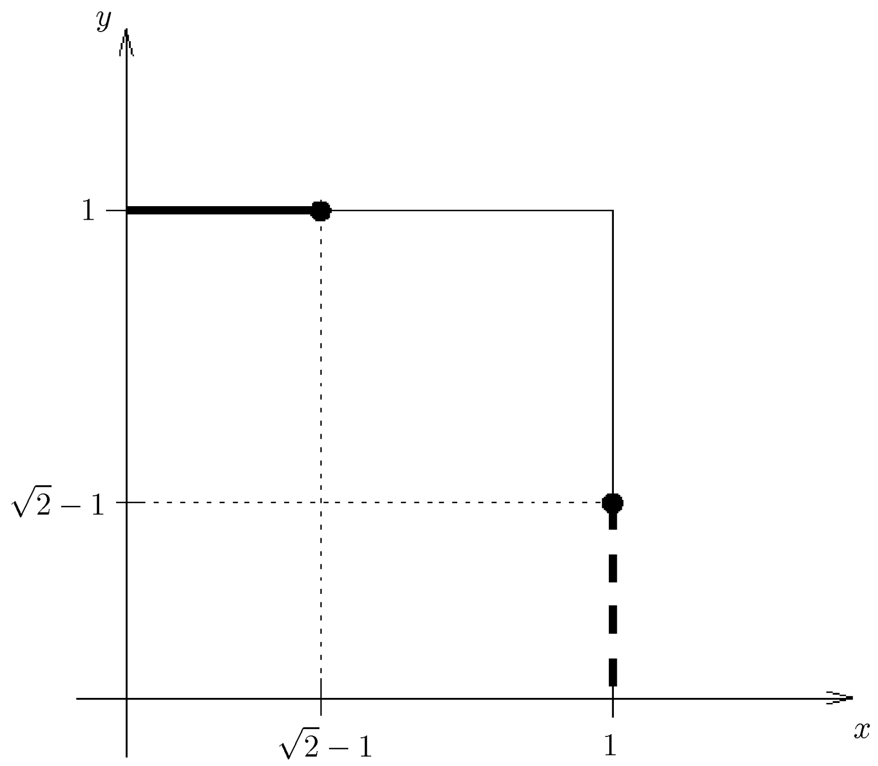



*$R_1(y)$ exists only if*

$$\begin{aligned} y^2 &\leq 1-2y \\ y^2+2y-1 &\leq 0 \\ y_{12} &= \frac{-2\pm\sqrt{4+4}}{2} = -1\pm\sqrt{2} = 0.4142 \textit{ or } -2.4142 \\ y &\leq 0.4142 \end{aligned}$$




$$R_1(y) = \begin{cases} 1 & \textit{if } y \leq 0.4142 \\ \textit{does not exist,} & \textit{otherwise} \end{cases}$$




*No match, no equilibrium.*

$\nabla$

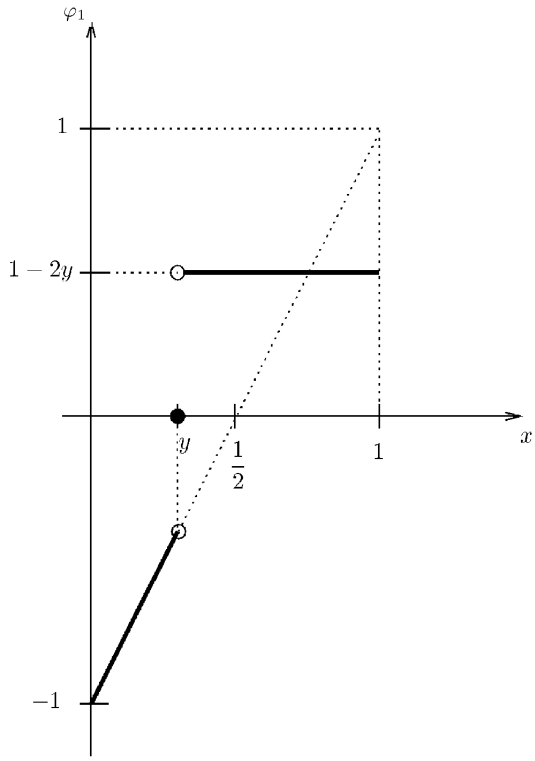
**41. Duel with sound**
Same as previous problem, but guns have no silencers:

$$\varphi_1 = \begin{cases} P_1(x) - (1 - P_1(x)) = 2P_1(x) - 1 & \text{if } x < y \\ P_1(x) - P_2(y) & \text{if } x = y \\ -P_2(y) + (1 - P_2(y)) = 1 - 2P_2(y) & \text{if } x > y \end{cases}$$

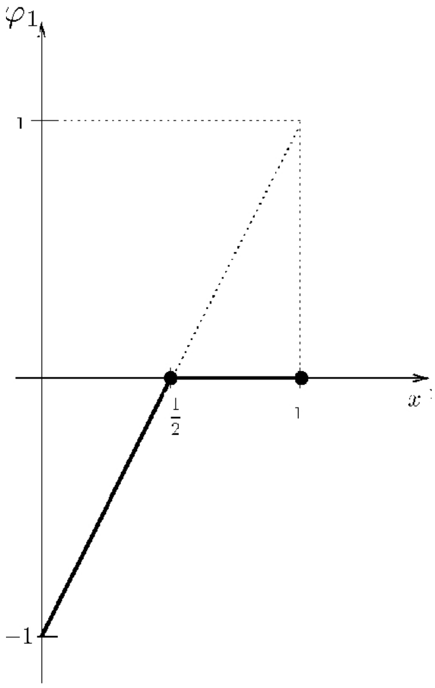
**Example 2.5** $P_1(x) = x, P_2(y) = y$

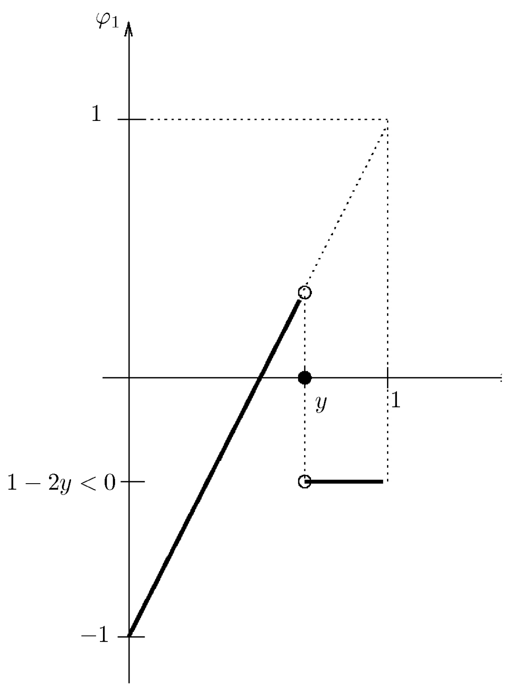
$$\varphi_1 = \begin{cases} 2x - 1 & \textit{if } x < y \\ 0 & \textit{if } x = y \\ 1 - 2y & \textit{if } x > y \end{cases}$$

*Case 1:* $y < \frac{1}{2}$

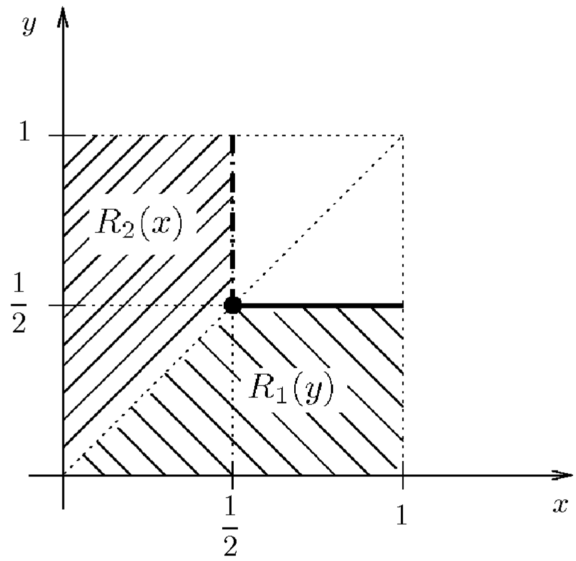



*Case 2:* $y = \frac{1}{2}$ *and Case 3:* $y > \frac{1}{2}$




$$R_1(y) = \begin{cases} \{x \mid x > y\} & \textit{if } y < \frac{1}{2} \\ \{x \mid x \geq \frac{1}{2}\} & \textit{if } y = \frac{1}{2} \\ \textit{does not exist} & \textit{if } y > \frac{1}{2} \end{cases}$$




*$R_2(x)$ is mirror image, only equilibrium is $x = y = \frac{1}{2}$*
$\Rightarrow$ *do not shoot early and also do not shoot late.* $\nabla$

**42. Spying game**

**Players:** Spy and counterespionage

**Strategies:** Efforts, $x, y \geq 0$

**Payoffs:** Let

$$\begin{aligned} P(x,y) &= \text{probability of arrest} \\ V(x) &= \text{value of information collected by spy} \\ U &= \text{value of spy.} \end{aligned}$$

Then

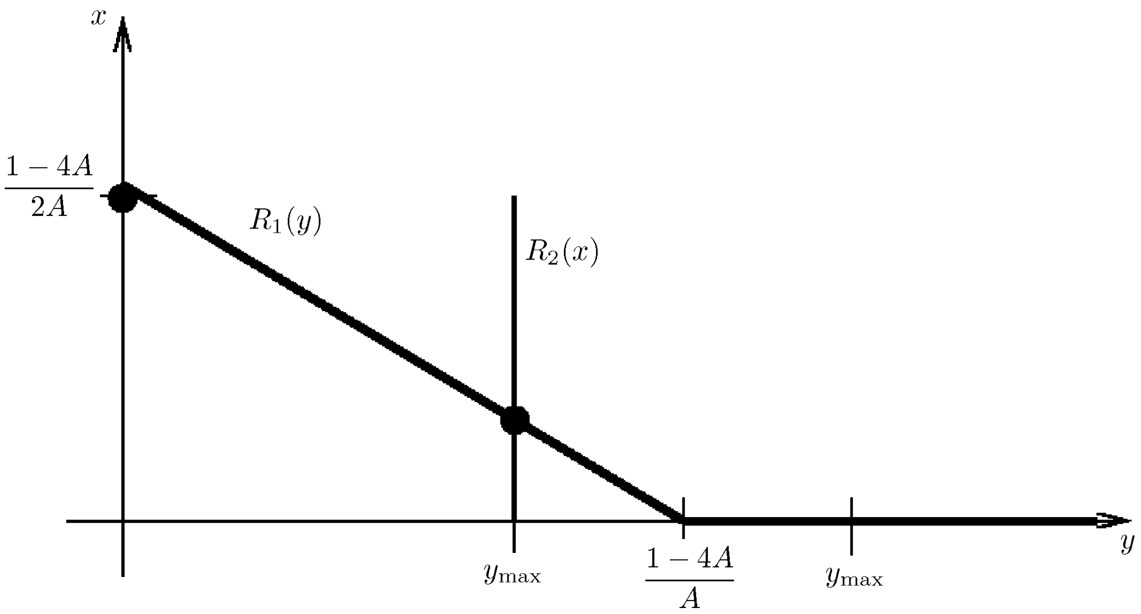
$$\begin{aligned} \varphi_1 &= P(x,y)\cdot(-U) + (1 - P(x,y))\cdot V(x) \\ \varphi_2 &= -\varphi_1 \end{aligned}$$

**Example 2.6** *$U = 4, V(x) = x, P(x,y) = A\cdot(x+y)$ ($A > 0$ is small)*

$$\begin{aligned} \varphi_1 &= A\cdot(x+y)(-4) + [1 - A\cdot(x+y)]x \\ &= -4Ax - 4Ay + x - Ax^2 - Axy \quad \textit{strictly concave in } x \\ \frac{\partial\varphi_1}{\partial x} &= -4A + 1 - 2Ax - Ay = 0 \\ x &= \frac{1-4A-Ay}{2A} \textit{ stationary point} \\ R_1(y) &= \begin{cases} \frac{1-4A-Ay}{2A} & \textit{if } y \leq \frac{1-4A}{A} \\ 0 & \textit{otherwise} \end{cases} \\ \varphi_2 &= 4Ax + 4Ay - x + Ax^2 + Axy \end{aligned}$$

*strictly inreases in y, so*

$$R_2(x) = y_{max}$$




*Equilibrium:*

$$y = y_{max}$$

$$x = \begin{cases} 0 & if\ y_{max} \geq \frac{1-4A}{A} \\ \frac{1-4A-Ay_{max}}{2A}, & otherwise \end{cases}$$

▽

**43. Hidden bomb in a city**

City with rectangular shape with $m$ horizontal and $n$ vertical blockrows and blockcolumns respectively. Value of block $(i, j)$ is $a_{ij}$.

City map:

| | | | | | | |
|---|---|---|---|---|---|---|
| $a_{11}$ | $a_{12}$ | $a_{13}$ | $\ldots$ | $a_{1,n-2}$ | $a_{1,n-1}$ | $a_{1n}$ |
| $a_{21}$ | $a_{22}$ | $a_{23}$ | $\ldots$ | $a_{2,n-2}$ | $a_{2,n-1}$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| $a_{m1}$ | $a_{m2}$ | $a_{m3}$ | $\ldots$ | $a_{m,n-2}$ | $a_{m,n-1}$ | $a_{mn}$ |

A terrorist group places a bomb in one of the blocks, and requests release of criminals from city prisons. City can check only one blockrow or blockcolumn to find the bomb.

**Players:** City (C) and terrorists (T)

**Strategies:**

For C: row $i$ or column $j$ to search

For T: where to place the bomb

**Payoffs:** $\varphi_1$ = value of block if the bomb was there and became found:

| | $1\backslash 2$ | (1,1) | $\ldots$ | $\ldots$ | (1,n) | (2,1) | $\ldots$ | $\ldots$ | (2,n) | $\ldots$ | (m,1) | $\ldots$ | $\ldots$ | (m,n) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| rows | 1 | $a_{11}$ | $\ldots$ | $\ldots$ | $a_{1n}$ | | | | | | | | | |
| | 2 | | | | | $a_{21}$ | $\ldots$ | $\ldots$ | $a_{2n}$ | | | | | |
| | $\vdots$ | | | | | | | | | $\ddots$ | | | | |
| | m | | | | | | | | | | $a_{m1}$ | $\ldots$ | $\ldots$ | $a_{mn}$ |
| columns | 1 | $a_{11}$ | | | | $a_{21}$ | | | | | $a_{m1}$ | | | |
| | 2 | | $a_{12}$ | | | | $a_{22}$ | | | | | $a_{m2}$ | | |
| | $\vdots$ | | | $\ddots$ | | | | $\ddots$ | | | | | $\ddots$ | |
| | n | | | | $a_{1n}$ | | | | $a_{2n}$ | | | | | $a_{mn}$ |

$$\varphi_1$$

$$\varphi_2 = -\varphi_1$$

Equilibrium: Matrix element is largest in its column and smallest in its row.

Facts:

largest elements in all columns are positive,

smallest elements in all rows are zeros

$\Rightarrow$ no element satisfies both $\Rightarrow$ no equilibrium

**44. First price auction**

One unit is sold in an auction

**Players:** $n$ potential buyers with subjective valuation of the unit $v_1 > v_2 > ... > v_n$ (known to all)

**Strategies:** Each of them presents a bid, $x_1, ..., x_n$, simultaneously, bids are secret

**Payoffs:** Highest bidder wins the unit and pays his price, in case of more highest bidders the one with highest valuation wins

$$\varphi_k = \begin{cases} v_k - x_k & \text{if } x_k = max\{x_1, ..., x_n\}\text{, and maximum is unique} \\ & \text{or } k = min\{l | x_l = x_k\} \\ 0 & \text{otherwise} \end{cases}$$

**Fact**: In any Nash equilibrium player 1 wins.

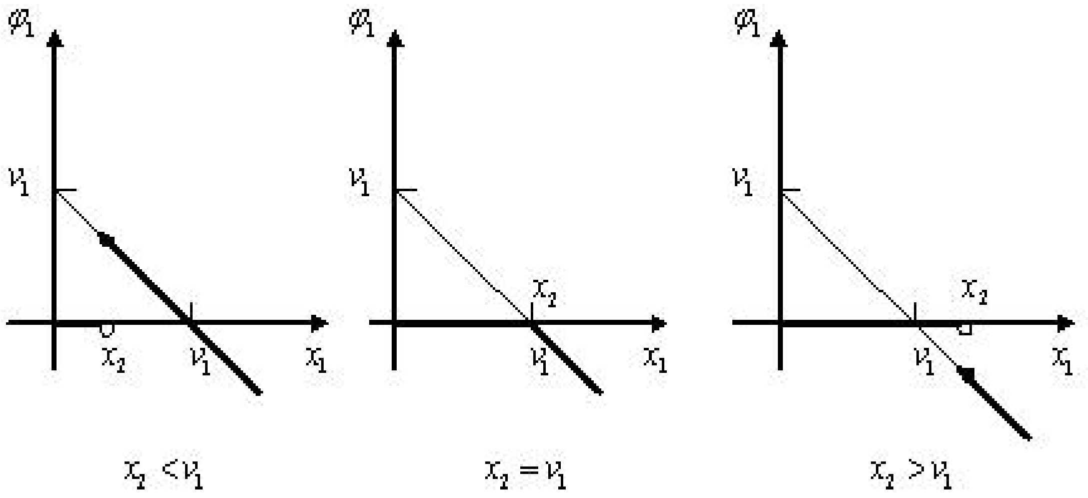
**Proof**: Assume not, if player $i \neq 1$ wins, then $x_i > x_1$. If $x_i > v_2$, then $\varphi_i = v_i - x_i \leq v_2 - x_i < 0$, so player $i$ can increase his payoff to zero by decreasing his bid $x_i$.

If $x_i \leq v_2$, then player 1 can increase his 0 payoff to $v_1 - x_i$ by increasing his bid to $x_i$. (Note, $v_1 - x_i > v_2 - x_i \geq 0$) ■

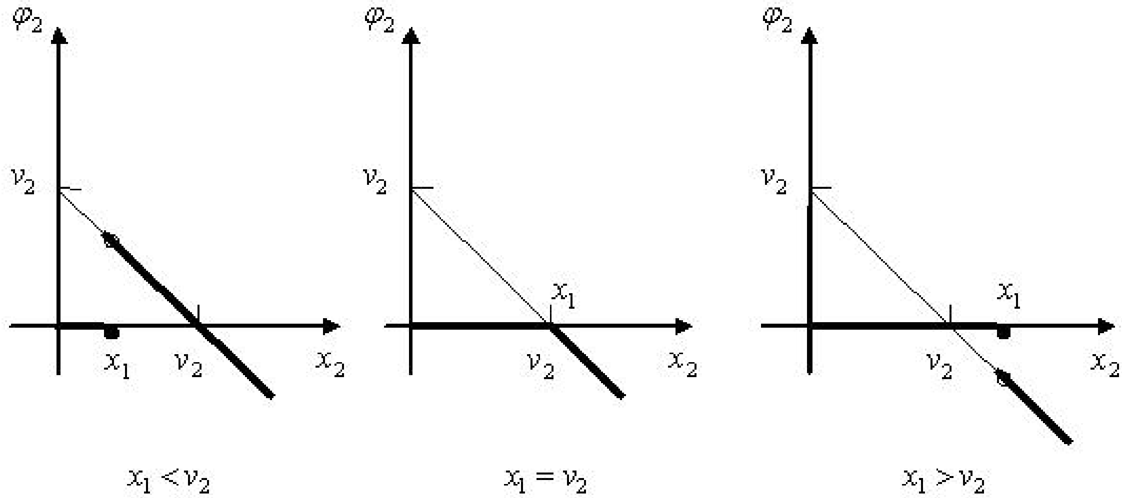
**Example 2.7** *n=2, payoffs:*

$$\varphi_1 = \begin{cases} v_1 - x_1 & \textit{if } x_1 \geq x_2 \\ 0 & \textit{otherwise} \end{cases}$$

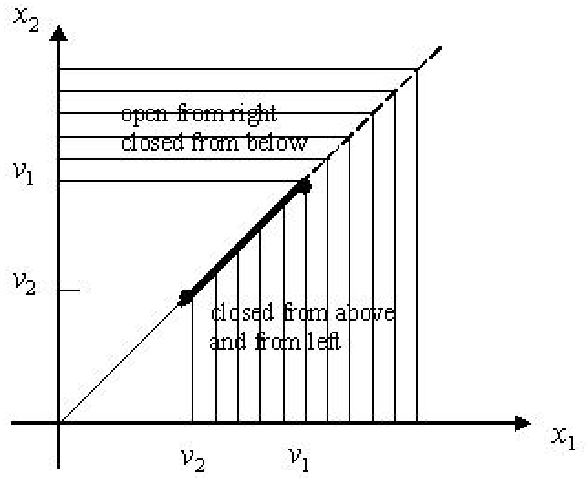
$$\varphi_2 = \begin{cases} v_2 - x_2 & \textit{if } x_2 > x_1 \\ 0 & \textit{otherwise} \end{cases}$$




$$R_1(x_2) = \begin{cases} x_2 & \textit{if } x_2 < v_1 \\ [0, x_2] & \textit{if } x_2 = v_1 \\ [0, x_2) & \textit{if } x_2 > v_1 \end{cases}$$




$$R_2(x_1) = \begin{cases} \emptyset & \textit{if } x_1 < v_2 \\ [0, x_1] & \textit{if } x_1 \geq v_2 \end{cases}$$




*Equilibrium set:* $\{(x_1, x_2)|v_2 \leq x_1 = x_2 \leq v_1\}$

**45. Second price auction**

**Players:** $n$ bidders with valuations $v_1 > v_2 > ... > v_n$

**Strategies:** Bids $x_1, x_2, ..., x_n$

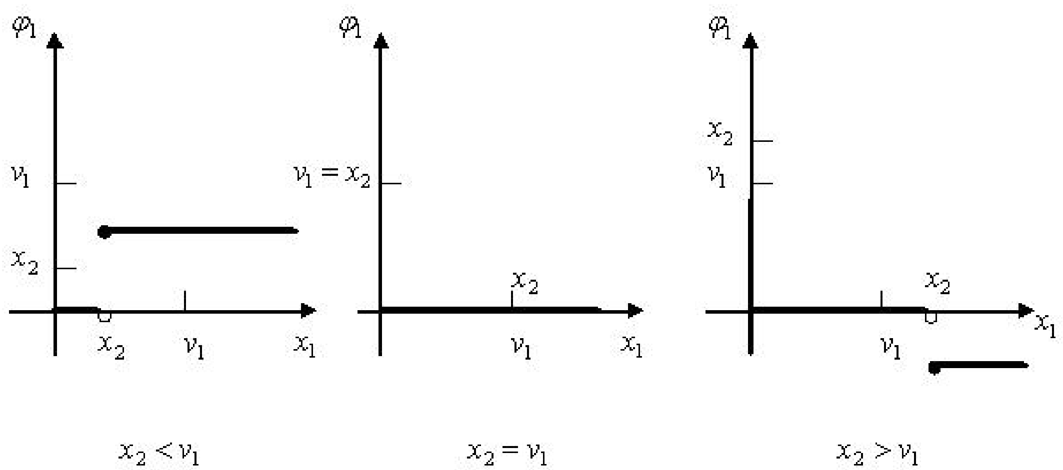
**Payoffs:** Player $k$ gets the item if he has the largest bid and in the case of a tie, his valuation is the largest; and the winner pays second largest bid

$$\varphi_k = \begin{cases} v_k - x_l & \text{if } x_k = max\{x_1, ..., x_n\}, x_l = max\{x_i|i \neq k\} \\ & \text{and } k = min\{i|x_i = x_k\} \text{ or unique maximal } x_k \\ 0 & \text{otherwise} \end{cases}$$

**Example 2.8** *n=2, payoffs:*

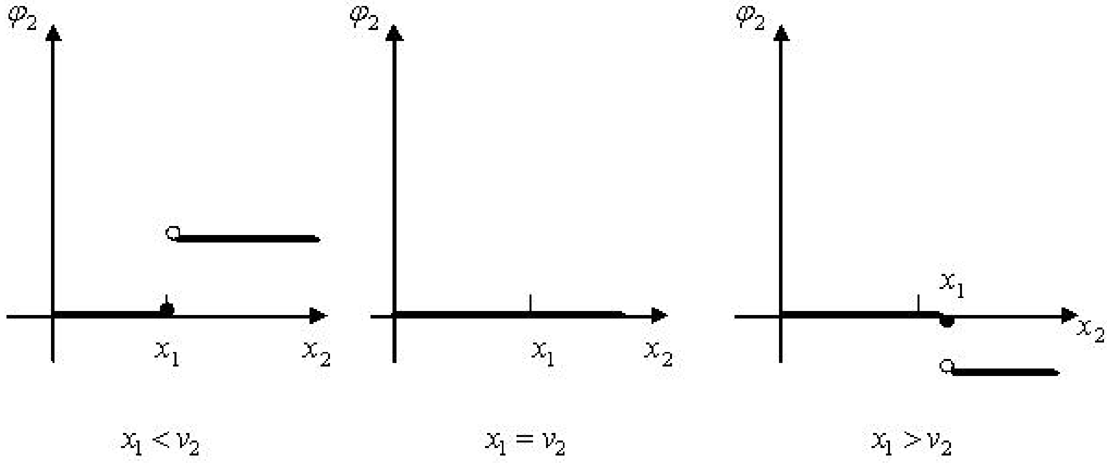
$$\varphi_1 = \begin{cases} v_1 - x_2 & \textit{if } x_1 \geq x_2 \\ 0 & \textit{otherwise} \end{cases}$$

$$\varphi_2 = \begin{cases} v_2 - x_1 & \textit{if } x_2 > x_1 \\ 0 & \textit{otherwise} \end{cases}$$



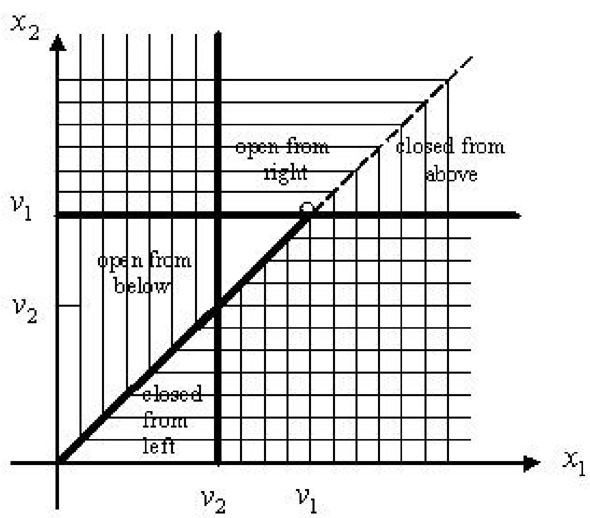
$$R_1(x_2) = \begin{cases} [x_2, \infty) & \textit{if } x_2 < v_1 \\ [0, \infty) & \textit{if } x_2 = v_1 \\ [0, x_2) & \textit{if } x_2 > v_1 \end{cases}$$

*(horizontally shaded region)*




$$R_2(x_1) = \begin{cases} (x_1, \infty) & \textit{if } x_1 < v_2 \\ [0, \infty) & \textit{if } x_1 = v_2 \\ [0, x_1] & \textit{if } x_1 > v_2 \end{cases}$$

*(vertically shaded region)*




*Set of equilibria= the two sets with both horizental and vertical shades.*

**46. Voting**

Two candidates ($A$ and $B$) run for an office which is decided by election. Among the voters $k$ support $A$, $m = n - k$ support $B$. Each vote costs amount $c$ ($0<c<1$) for the voter, so each of them makes the decision to vote or not.

**Players:** $n$ voters

**Strategies:** Votes ($x_i = 1$) or not ($x_i = 0$)

**Payoffs:** For voter, 1, 0, -1 if his candidate wins, the result is tie, or the opponent wins, but it decreases by $c$; for non voter, 1, 0 or -1 as above without voting cost

$1 \leq i \leq k$:

$$\varphi_i = \begin{cases} 1 - cx_i & \text{if } \sum_{l=1}^{k} x_l > \sum_{j=k+1}^{n} x_j \\ -cx_i & \text{if } \sum_{l=1}^{k} x_l = \sum_{j=k+1}^{n} x_j \\ -1 - cx_i & \text{if } \sum_{l=1}^{k} x_l < \sum_{j=k+1}^{n} x_j \end{cases}$$

$k + 1 \leq j \leq n$:

$$\varphi_j = \begin{cases} -1 - cx_j & \text{if } \sum_{i=1}^{k} x_i > \sum_{l=k+1}^{n} x_l \\ -cx_j & \text{if } \sum_{i=1}^{k} x_i = \sum_{l=k+1}^{n} x_l \\ 1 - cx_j & \text{if } \sum_{i=1}^{k} x_i < \sum_{l=k+1}^{n} x_l \end{cases}$$

**Example 2.9** *$k = m$=1*

| 1\ 2 | votes ($x_2 = 1$) | does not ($x_2 = 0$) |
|---|---|---|
| votes ($x_1 = 1$) | (-$c$, -$c$) | (1-$c$, -1) |
| does not ($x_1 = 0$) | (-1,1-$c$) | (0, 0) |

*Unique equilibrium ($x_1 = 1, x_2 = 1$)*

In general: What is the equilibrium?

**Fact**: One candidate wins, not an equilibrium

**Proof**:

If at least one voted in losing group $\Rightarrow$ if that voter does not vote, he increases payoff

If nobody voted in losing group, then two cases:

**(i)** more than one voted in winning group $\Rightarrow$ if one of them does not vote, he can increase payoff;

**(ii)** only one voted in winning group $\Rightarrow$ if one in losing group votes, then there is a tie, so he can increase his payoff.

$\Rightarrow$ At any equilibrium it has to be a tie. If somebody did not vote, then by changing his mind and voting, group becomes winner, so this is not equilibrium;

$\Rightarrow$ Everybody has to vote, so equilibrium exists only if $k = m$ and everybody votes for his candidate. This is really an equilibrium, since if any player does not vote, his group becomes the loser.

**47. Irrigation system**

Farms use common water supply to irrigate.

**Players:** $n$ farms

**Strategy:** Amount of water used, $x_1, \dots, x_n$

**Payoffs:** Benefit of irrigation – cost of water

$$\varphi_k = B_k(x_k) - x_k \cdot \underbrace{\frac{K\Big(\sum_{l=1}^{n} x_l\Big)}{\sum_{l=1}^{n} x_l}}_{\text{unit cost of water}}$$

Similar to oligopoly, $p(s) = -\frac{K(s)}{s}, C_k(x_k) = -B_k(x_k)$.

**48.Waste water management**
$n$ firms treat waste water in a common plant.

**Players:** $n$ firms

**Strategy:** Amounts of treated waste water, $x_1, \dots, x_n$

**Payoffs:** Benefit (usage of treated water, not paying penalty, etc.) – cost of water treatment

$$\varphi_k = B_k(x_k) - x_k \cdot \frac{K\Big(\sum_{l=1}^{n} x_l\Big)}{\sum_{l=1}^{n} x_l}.$$

Same as previous example.

**49. Multipurpose water management system**
$n$ water users (industry, agriculture, domestic, recreation).

**Players:** Water users

**Strategies:** Amounts of allocated water to users, $x_1, \dots, x_n$

**Payoffs:** Benefit – cost, same as above

**50. Chess game**

**Players:** 2 players controlling $W$ and $B$ figures

**Strategies:** For all possible configurations on the board a selected next move

**Payoffs:**

$$\begin{aligned} \varphi_W &= \begin{cases} 1 & \text{if } W \text{ wins} \\ -1 & \text{if } B \text{ wins} \\ 0 & \text{if tie.} \end{cases}, \\ \varphi_B &= -\varphi_W. \end{aligned}$$