Petit précis of games in strategic form

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Version 1.13

Autumn 2011

This typoscript resumes some important notions and results for games in strategic form. Below the ♠'s concern results. For proofs of these results we refer to the litterature. .

1 General notations

For $\mathbf{x} = (x^1, \ldots, x^n), \mathbf{y} = (y^1, \ldots, y^n) \in \mathbb{R}^n$ we write

$$\mathbf{x} \geq \mathbf{y} \text{ if } x^i \geq y^i \ (1 \leq i \leq n),$$

$$\mathbf{x} > \mathbf{y} \text{ if } \mathbf{x} \geq \mathbf{y} \text{ and } \mathbf{x} \neq \mathbf{y},$$

$$\mathbf{x} \gg \mathbf{y} \text{ if } x^i > y^i \ (1 \leq i \leq n).$$

With S_n we denote the group (under the composition operation) of permutations of the set $\{1, \ldots, n\}$.

2 Main notions

Definition 1 A game in strategic form (with $n \geq 1$ players) is an ordered $2n-$tuple

$$\Gamma = (X^1, \ldots, X^n; f^1, \ldots, f^n),$$

where, writing

$$\mathcal{N} = \{1, \ldots, n\},$$

the X^i are non-empty sets and where with

$$\mathbf{X} := X^1 \times \cdots \times X^n$$

the

$$f^i : \mathbf{X} \to \mathbb{R}$$

are functions. The elements of $\mathcal{N}$ are called *players*, X^i is called *strategy set* of player i and f^i is called *payof function* of player i. An element of X^i is called *strategy* of player i and an element of $\mathbf{x}$ is called *multi-strategy*.[1] If $\mathbf{x}$ is a multi-strategy, then $f^i(\mathbf{x})$ is called the *payoff* to player j at $\mathbf{x}$ and $(f^1(\mathbf{x}), \ldots, f^n(\mathbf{x}))$ is called *payoff vector* at $\mathbf{x}$.

Below we always denote by Γ a game in strategic form with n players, we identify $\mathbf{X}$ with $X^i \times \mathbf{X}^{\hat{\imath}}$, and accordingly write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x^i; \mathbf{x}^{\hat{\imath}})$. And for $\mathbf{x} \in \mathbf{X}$ we write

$$\mathbf{f}(\mathbf{x}) = (f^1(\mathbf{x}), \ldots, f^n(\mathbf{x})).$$

Definition 2 A game in strategic form

$$\Gamma = (X, \ldots, X; f^1, \ldots, f^N)$$

(with for each player the same strategy set X) is called *symmetric* if for each $\pi \in S_n,\ i \in \mathcal{N}$ and $\mathbf{x} \in \mathbf{X}$

$$f^i(x^1, \ldots, x^n) = f^{\pi(i)}(x^{\pi^{-1}(1)}, \ldots, x^{\pi^{-1}(n)}).$$

Definition 3 For $i \in \mathcal{N}$ and $\mathbf{z} \in \mathbf{X}^{\hat{\imath}}$ the conditional payoff function $f^i_{\mathbf{z}} : X^i \to \mathbb{R}$ is defined by

$$f^i_{\mathbf{z}}(x^i) := f^i(x^i; \mathbf{z}).$$

Definition 4 1. The *best-reply-correspondence* of player i the correspondence $R^i : \mathbf{X}^{\hat{\imath}} \multimap X^i$ defined by

$$R^i(\mathbf{z}) := \mathrm{argmax} f^i_{\mathbf{z}}.$$

$R^i(\mathbf{z})$ is called the *best-reply-set* of player i against $\mathbf{z}$.

[1] In the litterature also the term *strategy profile* is used instead of 'multi-strategy'.

2. The *best-reply-correspondence* of player i is the correspondence $\mathbf{R} : \mathbf{X} \multimap \mathbf{X}$ defined by

$$\mathbf{R}(\mathbf{x}) := R^1(\mathbf{x}^{\hat{1}}) \times \cdots \times R^n(\mathbf{x}^{\hat{n}}).$$

Definition 5 The *best-reply-payoff-function* of player i is the function $\phi^i : \mathbf{X}^{\hat{\imath}} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\phi^i(\mathbf{z}) = \sup_{x \in X^i} f^i_{\mathbf{z}}(x^i).$$

$\phi^i(\mathbf{z})$ is called the *best-reply-payoff* of player i at $\mathbf{z}$.

Definition 6 $\mathbf{x} \in \mathbf{X}$ is called an *nash equilibrium* of Γ if for every $i \in \mathcal{N}$ and $y^i \in X^i$

$$f^i(y^i; \mathbf{x}^{\hat{\imath}}) \leq f^i(\mathbf{x}).$$

We denote the set of nash equilibria of Γ by

$$E(\gamma).$$

Definition 7 1. $d^i \in X^i$ is called a *dominant strategy* of player i if

$$f^i(d^i; \mathbf{z}) \geq f^i(x^i; \mathbf{z})$$

for every $x^i \in X^i$ and $\mathbf{z} \in \mathbf{X}^{\hat{\imath}}$.

2. $d^i \in X^i$ is called a *strictlty dominant strategy* of player i if

$$f^i(d^i; \mathbf{z}) > f^i(x^i; \mathbf{z})$$

for every $x^i \in X^i \setminus \{d^i\}$ and $\mathbf{z} \in \mathbf{X}^{\hat{\imath}}$.

Definition 8 Let $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\boldsymbol{\lambda} > \mathbf{0}$. A multi-strategy$\mathbf{x}$ is called $\boldsymbol{\lambda}-$*weighted full cooperative* if it maximises the function

$$\sum_{j=1}^{n} \lambda^j f^j$$

In case $\boldsymbol{\lambda} = \mathbf{1}$ we call such a multi-strategy also *full cooperative*.

Definition 9 If $\mathbf{x}$ and $\mathbf{z}$ are multi-strategies, then one says:

- $\mathbf{z}$ is a *pareto-improvement* of $\mathbf{x}$ if $\mathbf{f}(\mathbf{z}) > \mathbf{f}(\mathbf{x})$;
- $\mathbf{z}$ is an *unanimous pareto-improvement* of $\mathbf{x}$ if $\mathbf{f}(\mathbf{z}) \gg \mathbf{f}(\mathbf{x})$.

A multi-strategy $\mathbf{x}$ is called

- *(strongly) pareto-efficiënt* if there does not exist a pareto-improvement of $\mathbf{x}$.
- *weakly pareto-efficiënt* if there does not exist an unanimous pareto-improvement of $\mathbf{x}$.

A multi-strategy $\mathbf{x}$ is called

- *(strongly) pareto-inefficient* if it is not pareto efficient.
- *(weakly) pareto-inefficient* if it is not weakly pareto efficient.

3 Dominant strategies

♠ **1** 1. *Each player has at most one strictly dominant strategy.*

2. *If d^j is a dominant strategy of player j, then his best-reply-payoff-function is given by $\phi^j(\mathbf{z}) = f^j(d^j; \mathbf{z})$.*

♠ **2** 1. *If each player j has a dominant strategy d^j, then the multi-strategy $\mathbf{d} := (d^1, \ldots, d^n)$ is a nash equilibrium. Such an nash equilibrium also is called a* dominant equilibrium.

2. *If each player j has a strictly dominant strategy d^j, then the multi-strategy $\mathbf{d} := (d^1, \ldots, d^n)$ is a nash equilibrium. This nash equilibrium also is called* strictly dominant equilibrium.

3. *If player j has a strictly dominant strategy d^j, then it holds for each nash equilibrium $\mathbf{e}$ that $e^j = d^j$.*

4 Best response correspondences and Nash equilibria

♠ **3** *The following statements for $\mathbf{x} \in \mathbf{X}$ are equivalent:*

1. *$\mathbf{x} \in \mathbf{X}$ is a nash equilibrium;*

2. *$x^j \in R^j(\mathbf{x}^{\hat{j}})$ $(j \in \mathcal{N})$.*

3. *$\mathbf{X}$ is a fixed point of $\mathbf{R}$.*

♠ **4** *If each strategy set is a metric space and each payoff function is continuous, then the set of nash equilibria is a closed subset of $\mathbf{X}$.*

5 Existence, semi-uniqueness and uniqueness of nash equilibria

♠ **5** *(Isoda-Nikaido) The following conditions together guarantee the existence of a nash equilibrium.*

1. *each strategy set X^i is a compact convex subset of a finite dimensional linear topological space;*

2. *each payoff function f^i is continuous;*

3. *the set of maximiser of each conditional payoff function $g^i_{\mathbf{z}}$ is convex.*[2]

♠ **6** *Suppose $\mathbf{X}$ is a metric space and the best-reply-correpondance $\mathbf{R} : \mathbf{X} \multimap \mathbf{X}$ is singleton-valued and a contraction. Then there exists at most one nash equilibrium. If $\mathbf{X}$ is complete, then there is a unique nash equilibrium.*

6 Pareto efficient multi-strategies

♠ **7** *Let $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\boldsymbol{\lambda} > \mathbf{0}$ and let $\mu > 0$. The set of $\boldsymbol{\lambda}-$weighted full cooperative multi-strategies and the set of $\mu\boldsymbol{\lambda}-$weighted full cooperative multi-strategies are the same.*

♠ **8** 1. *Let $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\boldsymbol{\lambda} > \mathbf{0}$. Each $\boldsymbol{\lambda}$-weighted full cooperative multi-strategy is weakly pareto-efficient.*

2. *Let $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\boldsymbol{\lambda} \gg \mathbf{0}$. Each $\boldsymbol{\lambda}$-weighted full cooperative multi-strategy is strongly pareto-efficient.*

[2] Sufficient for this is that each conditional payoff function is quasi-concave.

♠ **9** *Suppose $\mathbf{X}$ is a quasi-compact subset of a topological space, each payoff function f^j is continuous and $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\boldsymbol{\lambda} > \mathbf{0}$. Then there exists a $\boldsymbol{\lambda}$-weighted full cooperative multi-strategy.*

♠ **10** *If each strategy set is a convex subset of a linear space and each payoff function strictly strictly concave, then the set of weakly pareto efficient multi-strategies equals the set of strongly pareto efficient multi-strategies.*

♠ **11** *Suppose the strategy set of each player is a metric space and each payoff function is continuous. Then:*

1. *The set of weakly pareto efficient multi-strategies is closed.*
2. *If each strategy set is compact, then the set of strongly pareto efficiente multi-strategies is compact and not-empty.*
3. *If each strategy set is compact, then for each $\mathbf{x} \in \mathbf{X}$ there exists a pareto-efficient multi-strategy $\mathbf{z}$ with $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{z})$.*

♠ **12** *Suppose $n = 2$, each strategy set is a convex subset of linear space and a metric space. If each payoff function is concave, then the set of strongly pareto efficient multi-strategies is closed.*

♠ **13** *Suppose $\mathbf{X}$ is a subset of a linear space and $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\boldsymbol{\lambda} > \mathbf{0}$. If the function $\sum_{j=1}^{n} \lambda^j f^j$ is strictly quasi-concave, then there exists at most one $\boldsymbol{\lambda}$-weighted full cooperatieve multi-strategy.*

♠ **14** *Suppose $\mathbf{X}$ is a compact metric space. and each payoff function is continuous. Then there exists for each $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\boldsymbol{\lambda} > \mathbf{0}$ an $\boldsymbol{\lambda}$-weighted full cooperative multi-strategy.*

♠ **15** *Suppose each strategy set is a convex subset of linear space and each payoff function is concave. Then for every $\mathbf{x} \in \mathbf{X}$: $\mathbf{x}$ is weakly pareto efficient $\Leftrightarrow$ there exists $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\boldsymbol{\lambda} > \mathbf{0}$ such that $\mathbf{x}$ is $\boldsymbol{\lambda}$-weighted full cooperative.*

7 Dictator-multi-strategies

Definition 10 A multi-strategy $\mathbf{b}$ is called *dictator-multi-strategy* for player i if $\mathbf{b}$ is a maximiser of f^i.

♠ **16** *Each dictator-multi-strategy is weakly pareto-efficient.*

8 Prisoners' dilemma games

Definition 11 A game in strategic form is called a *prisoners' dilemma game* if each player has a strictly dominant strategy and the strictly dominant equilibrium is weakly pareto-inefficient.

9 Social welfare loss

Definition 12 The *social welfare loss* of a game in strategic form Γ with bounded payoff functions is defined as the number

$$\sup_{\mathbf{x} \in \mathbf{X}} \sum_{l=1}^{n} f^l(\mathbf{x}) - \sup_{\mathbf{e} \in E(\Gamma)} \sum_{l=1}^{n} f^l(\mathbf{e}).$$

10 Minimax and maximin

Definition 13 Fix $i \in \mathcal{N}$.

1. $\overline{v}^i := \inf_{\mathbf{y}\in\mathbf{X}^{\hat{\imath}}} \sup_{x\in X^i} f^i(x;\mathbf{y}) (= \inf_{\mathbf{y}\in\mathbf{X}^{\hat{\imath}}} \phi^i(\mathbf{y}))$ is called *minimax-payoff* of player i. And $\mathbf{m} \in \mathbf{X}^{\hat{\imath}}$ such that $\overline{v}^i = \sup_{x\in X^i} f^i(x;\mathbf{m})$ is called an *optimal punishment* for player i.

2. $\underline{v}^i := \sup_{x^i\in X^i} \inf_{\mathbf{z}\in\mathbf{X}^{\hat{\imath}}} f^i(x^i;\mathbf{z})$. is called the maximin-payoff of player i; And $p^i \in X^i$ such that $\underline{v}^i = \inf_{\mathbf{z}\in\mathbf{X}^{\hat{\imath}}} f^i(p^i;\mathbf{z})$. is called a maximin strategy of player i.

Definition 14 $\mathbf{w} \in \mathbb{R}^n$ is called *(strictly) individually rational* for player i if

$$w^i \geq \overline{v}^i \quad (w^i > \overline{v}^i)$$

and *(strictly) individual rational* if $\mathbf{w}$ is (strictly) individual rational for each players.

♠ **17** *For each nash equilibrium* $\mathbf{e}$ *the payoff vector* $\mathbf{f}(\mathbf{e})$ *is individually rational.*

♠ **18** *Each strong equilibrium is a nash equilibrium and is weakly pareto efficiënt.*

11 Symmetric games

♠ **19** *Suppose* Γ *is symmetric.*

1. *If* Γ *has a unique nash equilibrium* $\mathbf{e}$*, then each player has the same payoff at* $\mathbf{e}$ *and* $e^1 = \cdots = e^n$.

2. *If* Γ *has a unique full cooperative multi-strategy* $\mathbf{y}$ *then at this multi-strategy each plsyer has the same payoff and* $y^1 = \cdots = y^n$.

♠ **20** *Each symmetric game in strategic form with* $\#X = 2$ *has a nash equilibrium.*

12 Strong equilibria

Definition 15 A *coalition* is a subset of $\mathcal{N}$ and a coalition structure is a sequence

$$\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$$

consisting of disjoint non-empty coalitions whose union is $\mathcal{N}$.

Notation: for a coalition S we define $\hat{S} := \mathcal{N} \setminus S$. If S is a non-empty coalition, then we define, with $\#S$ the number of elements of S, $\lambda_1(S), \ldots, \lambda_{\#S}(S)$ as the unique elements of $\mathcal{N}$ for which $\lambda_1(S) < \cdots < \lambda_{\#S}(S)$, $\quad S = \{\lambda_1(S), \ldots, \lambda_{\#S}(S)\}$ and, using this notation,

$$\mathbf{X}^S := X^{\lambda_1(S)} \times \cdots \times X^{\lambda_{\#S}(S)}.$$

We identify $\mathbf{X}$ with $X^S \times X^{\hat{S}}$ and write according to this identification $\mathbf{x} \in \mathbf{X}$ als $\mathbf{x} = (\mathbf{x}^S; \mathbf{x}^{\hat{S}})$.

Definition 16 A multi-strategy $\mathbf{x}$ is called a *strong (nash) equilibrium* of Γ if there does not exist a non-empty coalition S and $\mathbf{y} \in \mathbf{X}^S$ such that $f^i(\mathbf{y}; x^{\hat{S}}) > f^i(\mathbf{x})$ $(i \in S)$.

Referenties

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