Uncertainty

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Utility maximisation

Three main types for setting of utility maximisation:

- standard;
- intertemporal;
- with uncertainty.

Motivation

It is not true that a consumer lways prefers outcomes with a higher expected money value:

- Getting 100.000 Euro with chance $1/80$ has expected value 1250.
- Getting 1000 Euro for sure has expected value 1000.

Motivation

Other example: Saint-Petersbourgh paradox (Bernouilli). Suppose a fair coin (with a heads and tails side) is tossed until it comes up heads. Your payoff depends on the number of tosses before heads appears for the first time. When this number is n, You receive 2^n Euro.

Expected value

$$= \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + \cdots = +\infty.$$

How much would You pay to participate in this game?

These examples suggest that we need some concept other than expected (money) value to analyze how people make decisions in risky situations. Expected utility is such a concept.

Setting

Consider a decision maker with uncertainty about the outcomes of given choices he can made. These choices concern gambles.

Gamble $' =' $ lottery ticket. Notation: $((c_1, \pi_1), \ldots, (c_n, \pi_n))$. Interpretation: there are n possible outcomes. The gamble leads to outcome c_j with probability π_j; so $\pi_1 + \cdots + \pi_n = 1$.

Some gambles, lead to outcomes for sure; denote the gamble that leads to outcome c for sure by $(c, 1)$ or just by c.

Gambles

- There is a set of outcomes that themselves involve no uncertainty.
- (Simple) gamble: assigns a probability to each outcome.
- Gamble is monetary if outcomes are amounts of money. (Negative amount means a loss.)
- Compound gamble: outcomes are themselves (simple or not simple) gambles. (We will not deal with.)

Von-Neumann-Morgenstern utility

Assumption: the decision maker has preference relation $\succsim$ over the set of gambles and makes choices according to this relation.

- Under reasonable conditions there exists a utility function U (defined on the set of gambles) representing $\succsim$.
- There even exists one with the expected utility property, i.e. it assigns to each gamble the expected value of the utilities that might result. I.e.

$$U((c_1, \pi_1), \ldots, (c_n, \pi_n)) = \pi_1 U(c_1) + \cdots + \pi_n U(c_n).$$

Such utility function with the expected utility property is called a 'Von-Neumann-Morgenstern utility function'.

Von-Neumann-Morgenstern utility

- Thus when we know the value of von-Neumann-Morgenstern function on all sure outcomes, then we know it on all gambles.
- Von-Neumann-Morgenstern utility functions are unique op to positive affine transformations. (Utility here is cardinal!)

Monetary outcomes

From now on we assume that outcomes are monetary.

Given a Von-Neumann-Morgenstern utility function U and a gamble $((c_1, \pi_1), \dots, (c_n, \pi_n))$ we have the following notions:

- $\pi_1 c_1 + \cdots + \pi_n c_n$: expected value of wealth.
- $\pi_1 U(c_1) + \cdots + \pi_n U(c_n)$: expected utility of wealth.
- $U(\pi_1 c_1 + \cdots + \pi_n c_n)$: utility of expected value of wealth.

Exemple

Consumer with 1000 yen can participate in a gamble that gives a 50 percent probability of winning 500 yen and a 50 percent probability of loosing 500 yen. This leads to gamble:

$$((1500 \text{ yen}, 1/2), (500 \text{ yen}, 1/2)).$$

- Expected value of wealth: $\frac{1}{2} \cdot 1500 + \frac{1}{2} \cdot 500 = 1000$ yen. (So same value then with not participating.)
- Expected utility of wealth: $\frac{1}{2}U(1500) + \frac{1}{2}U(1500)$.
- Utility of expected value of wealth: $U(1000)$.

Interpretation: $U(1000) > \frac{1}{2}U(1500) + \frac{1}{2}U(1500)$: risk averse.
$U(1000) = \frac{1}{2}U(1500) + \frac{1}{2}U(1500)$: risk neutral.
$U(1000) < \frac{1}{2}U(1500) + \frac{1}{2}U(1500)$: risk lover.

Types of risk

- Risk averse:
  $U(\pi_1 c_1 + \cdots + \pi_n c_n) > \pi_1 U(c_1) + \cdots + \pi_n U(c_n)$.
  (Concave U.)
- Risk neutral:
  $U(\pi_1 c_1 + \cdots + \pi_n c_n) = \pi_1 U(c_1) + \cdots + \pi_n U(c_n)$.
  (Affine U.)
- Risk loving:
  $U(\pi_1 c_1 + \cdots + \pi_n c_n) < \pi_1 U(c_1) + \cdots + \pi_n U(c_n)$.
  (Convex U.)

Example

Joep has 100 euro. He can take a card from a (standard) card game.

- If he gets diamonds, then he obtains 20 euro and thus has 120 euro (outcome 1). Otherwise, he has to pay 20 euro and thus has 80 euro (outcome 2).
- His Von-Neumann-Morgenstern utility function for sure outcome c is given by $U(c) = c^2$. So

$$U((c_1, \pi_1), (c_2, \pi_2)) = \pi_1 c_1^2 + \pi_2 c_2^2.$$

- We have $\pi_1 = \frac{1}{4}, \pi_2 = \frac{3}{4}$.

Example

Will Joep participate in the gamble?

- Yes means: gamble $((120, \frac{1}{4}), (80, \frac{3}{4}))$.
- No means: gamble $((100, \frac{1}{4}), (100, \frac{3}{4}))$.
- $U((120, \frac{1}{4}), (80, \frac{3}{4})) = 8400$ and $U((100, \frac{1}{4}), (100, \frac{3}{4})) = 10.000$.
- Thus he will not participate. (Nevertheless Joep is risk loving.)