The Generalised Nikaido-Isoda Theorem

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Preface

The main contribution of this typoscript is a variant of a theorem in [6] of Nikaido and Isoda. I refer to it as the "Generalized Nikaido-Isoda Theorem". The variant is proven by considering, what if refer to as "best-response supersets" and "domination sets". In addition to the Generalized Nikaido-Isoda Theorem the typoscript contains various related results wherein such sets play a role. These results may be useful when dealing with Nash equilibria of games in strategic form where strategy sets may be not compact. No result here should be considered as new. Giving appropriate references is difficult.

Of course, some prior knowledge is required: what that entails will quickly become clear when browsing through the typoscript.

The typoscript can be downloaded at `http://pvmouche.deds.nl` (if that URL is still valid).[1] It is primarily intended for the scientific community and may be used at the own discretion. But it would be a sign of good character if the ⓒ was respected.

The author, of course, welcomes further comments and suggestions that may lead to improvements.

[1]The (idealistic) public domain typesetting system LaTeX was used for its production under the(idealistic) operating system Linux.

1 Setting

Let us quickly recall basic game theoretic terminology. We deal with n player games in strategic form where $N := \{1, \ldots, n\}$ is the set of players and for all $i \in N$, X_i is player i's strategy set and f_i is player i's payoff function. Henceforth, X_i is non-empty and

$$f_i : \mathbf{X} \to \mathbb{R}$$

is a function where $\mathbf{X} := X_1 \times \cdots \times X_n$. We denote such a game also by $(X_1, \ldots, X_n; f_1, \ldots, f_n)$.

For $i \in N$, let $\mathbf{X}_{\hat{\imath}} := X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. We sometimes identify $\mathbf{X}$ with $X_i \times \mathbf{X}_{\hat{\imath}}$ and then write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x_i; \mathbf{x}_{\hat{\imath}})$.

For a player $i \in N$ and $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$, i.e. strategy profile $\mathbf{z}$ of the other players, denote by $f_i^{(\mathbf{z})}$, the conditional payoff function of player i against $\mathbf{z}$, i.e. the function $f_i^{(\mathbf{z})} : X_i \to \mathbb{R}$ given by

$$f_i^{(\mathbf{z})}(x_i) \ := \ f_i(x_i; \mathbf{z}).$$

For $i \in N$, the correspondence $R_i \colon \mathbf{X}_{\hat{\imath}} \multimap X_i$ is defined by

$$R_i(\mathbf{z}) \ := \ \operatorname{argmax} f_i^{(\mathbf{z})}.$$

The joint best-response correspondence is the correspondence $\mathbf{R} : \mathbf{X} \multimap \mathbf{X}$ defined by

$$\mathbf{R}(\mathbf{x}) := R_1(\mathbf{x}_{\hat{1}}) \times \cdots \times R_n(\mathbf{x}_{\hat{n}})$$

Further, let for $i \in N$

$$R_i(\mathbf{X}_{\hat{\imath}}) = \cup_{\mathbf{z} \in \mathbf{X}_i} R_i(\mathbf{z})$$

be the set of best-responses of player i. When X_i is a metric space, the correspondence R_i is called bounded if the set $R_i(\mathbf{X}_{\hat{\imath}})$ is a bounded subset of X_i.

Denote by E the Nash equilibrium set, i.e. the set of $\mathbf{x} \in \mathbf{X}$ with the property that for all $i \in N$, x_i is a maximiser of $f_i^{(\mathbf{x}_{\hat{\imath}})}$. Also: E is the set of fixed points of $\mathbf{R}$. If the game is Γ, then we also denote this set by $E(\Gamma)$.

A strategy $x_i \in X_i$ is said to be strongly dominated by a strategy $d_i \in X_i$ if $f_i^{(\mathbf{z})}(d_i) > f_i^{(\mathbf{z})}(x_i)$ for each $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$.

Finally, given a correspondence $F : A \multimap B$, F is <u>proper</u> if $\#F(a) \geq 1$ for all $a \in A$. And if F is single-valued, i.e. $\#F(a) = 1$ for all $a \in A$, then we also interpret F as a mapping $F : A \to B$ by identifying for every $a \in A$ the set $F(a)$ with its unique element.

2 Maximisers of a restricted function

The next lemma presents useful simple results about the relation of the set of maximisers of the restriction of a function and the set of the maximisers of the unrestricted function.

`Leme:maximarestrict`

Lemma 1 *Suppose $g : X \to \mathbb{R}$ is a function and W is a non-empty subset of X. Then:*

1. (a) $[a \in \operatorname{argmax} g \ \wedge a \in W] \ \Rightarrow \ a \in \operatorname{argmax} g\restriction_W$.

   (b) *Sufficient for*

   $$\operatorname{argmax} g \subseteq \operatorname{argmax} g\restriction_W$$

   to hold is that $\operatorname{argmax} g \subseteq W$.

   (c) *Sufficient for*

   $$\operatorname{argmax} g = \operatorname{argmax} g\restriction_W$$

   to hold is that $\emptyset \neq \operatorname{argmax} g \subseteq W$.[2]

[2]The assumption $\emptyset \neq \operatorname{argmax} g$ is important: $g(x) = x$ ($x \in \mathbb{R}$), $W = [0, 137]$.

2. *Suppose* $\sup g\restriction(W \setminus X) \leq \sup g\restriction W$, *i.e.* $g(x) \leq \sup g\restriction W$ $(x \in X \setminus W)$. *Then:*

   (a) $\operatorname{argmax} g\restriction W \subseteq \operatorname{argmax} g$.

   (b) *If* $\operatorname{argmax} g\restriction W \neq \emptyset$, *then each of the following three conditions separately is sufficient for* $\operatorname{argmax} g = \operatorname{argmax} g\restriction W$ *to hold:*

      i. $\#\operatorname{argmax} g = 1$.

      ii. $g(x) < \sup g\restriction W$ $(x \in X \setminus W)$.

      iii. $\sup g\restriction(W \setminus X) < \sup g\restriction W$. $\diamond$

Proof.— 1a. Suppose $a \in \operatorname{argmax} g$ and $a \in W$. So $g(x) \leq g(a)$ $(x \in X)$. Thus also $g(x) \leq g(a)$ $(x \in W)$. As $a \in W$, $a \in \operatorname{argmax} g\restriction W$ follows.

1b. By part 1a.

1c. By part 1b, it is sufficient to prove "$\supseteq$". So suppose $b \in \operatorname{argmax} g\restriction W$. So $b \in W$ and $g(x) \leq g(b)$ $(x \in W)$. By assumption, $\operatorname{argmax} g \neq \emptyset$. Fix $a \in \operatorname{argmax} g$. As, also by assumption, $a \in W$, we have $g(a) \leq g(b)$. As $a \in \operatorname{argmax} g$, we have $g(b) \leq g(a)$. Thus $g(b) = g(a)$ and therefore also $b \in \operatorname{argmax} g$.

2a. Suppose $a \in \operatorname{argmax} g\restriction W$, so a is a maximiser of $g\restriction W$ and therefore $\max g\restriction W = \sup g\restriction W = g(a)$. As for every $x \in X \setminus W$, also $g(x) \leq \sup g\restriction(X \setminus W) \leq \sup g\restriction W = g(a)$, a even is a maximiser of g.

2bi. The proof is complete by part 1a if we show that $\operatorname{argmax} g \subseteq W$. This we do by contradiction. So suppose $\operatorname{argmax} g \not\subseteq W$. As $\#\operatorname{argmax} g = 1$, $\operatorname{argmax} g$ is a singleton, say x. Fix $a \in \operatorname{argmax} g\restriction W$. Now $\sup g\restriction W = g(a)$. As $\{x\} \not\subseteq W$, we have $x \in X \setminus W$ and therefore, $g(x) \leq \sup g\restriction(X \setminus W) \leq \sup g\restriction W = g(a)$. Of course, also $g(x) \geq g(a)$. So $g(x) = g(a)$. As x is the unique maximiser of g, we have $x = a \in W$, a contradiction.

ii. The proof is complete by part 1a if we show that $\operatorname{argmax} g \subseteq W$. This we do by contradiction. So suppose $\operatorname{argmax} g \not\subseteq W$. Fix $x \in \operatorname{argmax} g$ with $x \notin W$. Now $g(x) < \sup g\restriction W$, which is absurd.

iii. By part 2bii. $\square$

3 Best-response supersets and domination sets

Defe:bestrespposeset

Definition 1 *Consider a game in strategic form. Let* $i \in N$. *A subset* K_i *of* X_i *is a* *<u>best-response superset</u>* *for player* i *if* $R_i(\mathbf{X}_{\hat{i}}) \subseteq K_i$. $\diamond$

Note that X_i and $R_i(\mathbf{X}_{\hat{i}})$ are best-response supersets for player i and that the empty set also may be a best-response superset. Also note that if K_i is a best-response superset for player i and $K'_i \supseteq K_i$, then also K'_i is a best-response superset for that player.

Proe:zelfdebr

Proposition 1 *Consider a game in strategic form. Let* K_i *be a best-response superset for player* i *and* $\mathbf{z} \in \mathbf{X}_{\hat{i}}$. *If* $R_i(\mathbf{z}) \neq \emptyset$, *then* $R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})} = \operatorname{argmax} f_i^{(\mathbf{z})}\restriction K_i$. $\diamond$

Proof.— Suppose $R_i(\mathbf{z}) \neq \emptyset$, so $\operatorname{argmax} f_i^{(\mathbf{z})} \neq \emptyset$. Lemma 1(1c) implies the desired result as $\emptyset \neq \operatorname{argmax} f_i^{(\mathbf{z})} \subseteq K_i$. $\square$

3.1 Domination sets

Defe:effcap

Definition 2 *Consider a game in strategic form. Let* $i \in N$. *A subset* K_i *of* X_i *is a* *<u>domination set</u>* *for player* i *if* $K_i \neq \emptyset$ *and every* $x_i \in X_i \setminus K_i$ *is strongly dominated by some element of* K_i. $\diamond$

Note that X_i is domination set. Also note that if K_i is a domination set for player i and $K'_i \supseteq K_i$, then also K'_i is a domination set for that player.

Lemma 2 *Consider a game in strategic form. Let $i \in N$ Suppose K_i is a domination set for player i. Then for all $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ and $x_i \in X_i \setminus K_i$ it holds that $f_i^{(\mathbf{z})}(x_i) < \sup f_i^{(\mathbf{z})} \upharpoonright K_i$.* $\diamond$

Proof.— Suppose $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ and $x_i \in X_i \backslash K_i$. By definition, x_i is strongly dominated by some element of K_i, say by d_i. This implies $f_i^{(\mathbf{z})}(x_i) < f_i^{(\mathbf{z})}(d_i)$. It follows that $f_i^{(\mathbf{z})}(x_i) < f_i^{(\mathbf{z})}(d_i) \leq \sup f_i^{(\mathbf{z})} \upharpoonright K_i$. $\square$

`Proe:domi`

Proposition 2 *Consider a game in strategic form. Let $i \in N$. Suppose K_i is a domination set for player i. Then $R_i(\mathbf{X}_{\hat{\imath}}) \subseteq K_i$, i.e. K_i is a best-response superset for player i.* $\diamond$

Proof.— Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. We show by contradiction that $R_i(\mathbf{z}) \subseteq K_i$. So suppose $a_i \in R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})}$ with $a_i \notin K_i$. As K_i is a domination set for player i, a_i is strongly dominated by some $d_i \in K_i$. So $f_i^{(\mathbf{z}')}(d_i) > f_i^{(\mathbf{z}')}(a_i)$ for all $\mathbf{z}' \in \mathbf{X}_{\hat{\imath}}$. In particular $f_i^{(\mathbf{z})}(d_i) > f_i^{(\mathbf{z})}(a_i)$, which is a contradiction with a_i being a maximiser of $f_i^{(\mathbf{z})}$. $\square$

`Proe:zelfdebr2`

Proposition 3 *Consider a game in strategic form. Let $i \in N$, K_i a domination set for i and $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. Then $R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})} = \operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright K_i$.* $\diamond$

Proof.— By Proposition 2, K_i is a best-response superset for player i. So if $R_i(\mathbf{z}) \neq \emptyset$, then the desired result follows from Proposition 1. Now suppose $R_i(\mathbf{z}) = \emptyset$. By Lemma 1(2a), also $\operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright K_i = \emptyset$ and thus also in this case the desired result holds. $\square$

Note that in case X_i is a metric space player i has a bounded best-response superset if and only if the best-response correspondence R_i is bounded.

`Leme:compactsuperset`

Lemma 3 *Consider a game in strategic form. Let $i \in N$ and suppose the strategy set X_i is a closed subset of a finite dimensional normed real linear space.*

1. *Suppose player i has a bounded best-response superset. Then*
   (a) *Player i has a compact best-response superset.*
   (b) *If X_i is convex, then player i has a convex compact best-response superset.*
2. *Suppose player i has a bounded domination set. Then*
   (a) *Player i has a compact domination set.*
   (b) *If X_i is convex, then player i has a convex compact domination set.* $\diamond$

Bewijs.— Let E be the linear space.

1a. Let W_i be a bounded best-response superset.. As W_i is a bounded subset of X_i, W_i is a bounded subset of E. Let K_i be its topological closure. Also K_i is bounded. As E is finite dimensional, it follows that K_i is compact. As $W_i \subseteq X_i$ and X_i is closed, also K_i is a subset of X_i. As K_i contains W_i, it follows that K_i is a compact best-response superset for player i.

1b. Suppose X_i is convex. By part 1a, let K_i be a compact best-response superset. As E is finite dimensional, this implies that also $\mathrm{Conv}(K_i)$ is compact. As $K_i \subseteq X_i$ and X_i is convex, also $\mathrm{Conv}(K_i)$ is a subset of X_i. So $\mathrm{Conv}(K_i)$ is a convex compact best-response superset.

2a. Now player i has a bounded best-response superset, say L_i. As L_i is a bounded subset of X_i, L_i is a bounded subset of E. Let K_i be its topological closure. Also K_i is bounded. As E is finite dimensional, it follows that K_i is compact. As $L_i \subseteq X_i$ and X_i is closed, also L_i is a subset of X_i. As K_i contains L_i, it follows that K_i is a compact domination set for player i.

2b. Suppose X_i is convex. By part 2a, let K_i be a compact domination set. As E is finite dimensional, this implies that also $\mathrm{Conv}(K_i)$ is compact. As $K_i \subseteq X_i$ and X_i is convex, also $\mathrm{Conv}(K_i)$ is a subset of X_i. So $\mathrm{Conv}(K_i)$ is a convex compact best-response superset. $\square$

Leme:nonempty

Lemma 4 *Consider a game in strategic form where each strategy set is a convex subset of a normed real linear space. Let $i \in N$. Suppose that for player i the conditional payoff functions are continuous and that player i has a compact domination set. Then*

1. *The best-response correspondence R_i is proper.*
2. *Suppose that the conditional payoff functions of player i are strictly quasi-concave. Then R_i is single-valued.* $\diamond$

Proof.— 1. Let K_i be a compact domination set. By Proposition 3, $R_i(\mathbf{z}) = \mathrm{argmax} f_i^{(\mathbf{z})} \upharpoonright K_i$ ($\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$). Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. As also $f_i^{(\mathbf{z})} \upharpoonright K_i$ is continuous and K_i is non-empty and compact, $\#R_i(\mathbf{z}) \geq 1$ follows.

2. Let L_i be compact domination set. As the linear space is finite dimensional, the convex hull $K_i := \mathrm{Conv}(X_i)$ is also compact. So K_i is a convex compact domination set. By Proposition 3, $R_i(\mathbf{z}) = \mathrm{argmax} f_i^{(\mathbf{z})} \upharpoonright K_i$ ($\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$). Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$). We show that $\#R_i(\mathbf{z}) = 1$. By part 1, $\#R_i(\mathbf{z}) \geq 1$. As also $f_i^{(\mathbf{z})} \upharpoonright K_i$ is strictly quasi-concave, also $\#R_i(\mathbf{z}) \leq 1$ follows. $\square$

4 Restricted games

Leme:Ezelfde

Lemma 5 *Consider a game in strategic form*

$$\Gamma = (X_1, \ldots, X_n; f_1, \ldots, f_n).$$

Suppose for every player i that K_i is a non-empty subset of X_i. Let Γ' be the game in strategic form

$$\Gamma' = (K_1, \ldots, K_n; g_1, \ldots, g_n)$$

where $g_i := f_i \upharpoonright \mathbf{K}$. Then $E(\Gamma) \cap \mathbf{K} \subseteq E(\Gamma')$. $\diamond$

Proof.— Suppose $\mathbf{e} \in E(\Gamma) \cap \mathbf{K}$, i.e. for every i we have $e_i \in K_i$ and e_i is a maximiser of $f_i^{(\mathbf{e}_{\hat{\imath}})}$. Lemma 1(1) guarantees that e_i also is a maximiser of $f_i^{(\mathbf{e}_{\hat{\imath}})} \upharpoonright K_i$. Thus $\mathbf{e} \in E(\Gamma')$. $\square$

The result in Lemma 5 becomes more nice if every K_i is a non-empty best-response superset:

Proe:Ezelfde

Proposition 4 *Consider a game in strategic form*

$$\Gamma = (X_1, \ldots, X_n; f_1, \ldots, f_n).$$

Suppose for every player i that K_i is a non-empty best-response superset. Let Γ' be the game in strategic form

$$\Gamma' = (K_1, \ldots, K_n; g_1, \ldots, g_n)$$

where $g_i := f_i \upharpoonright \mathbf{K}$. For $i \in N$, let R'_i be the best-response correspondence of player i in the game Γ'. Then:

1. *For every $i \in N$ and $\mathbf{z} \in \mathbf{K}_{\hat{\imath}}$: $R_i(\mathbf{z}) \neq \emptyset \Rightarrow R_i(\mathbf{z}) = R'_i(\mathbf{z})$.*
2. *$E(\Gamma) \subseteq E(\Gamma')$.*
3. *If each best-response correspondence R_i is proper, then $E(\Gamma) = E(\Gamma')$.* $\diamond$

Proof.— First note that, for $\mathbf{z} \in \mathbf{K}_{\hat{\imath}}$, $R'_i(\mathbf{z}) = \mathrm{argmax}\, g_i^{(\mathbf{z})} = \mathrm{argmax}\, (f_i \upharpoonright \mathbf{K})^{(\mathbf{z})} = \mathrm{argmax}\, f_i^{(\mathbf{z})} \upharpoonright K_i$.

1. Suppose $R_i(\mathbf{z}) \neq \emptyset$. By Proposition 1, $R_i(\mathbf{z}) = \mathrm{argmax} f_i^{(\mathbf{z})} \upharpoonright K_i = R'_i(\mathbf{z})$.

2. Suppose $\mathbf{e} \in E(\Gamma)$. We prove that $\mathbf{e} \in \mathbf{K}$, i.e. that $e_i \in K_i$ ($i \in N$) and then the proof is complete by Lemma 5. Well, fix i. Note that $e_i \in R_i(\mathbf{e}_{\hat{\imath}})$. As K_i is a best-response superset, $e_i \in K_i$ holds.

3. Suppose each correspondence R_i is proper. By part 2, we still have to prove "$\supseteq$". So suppose $\mathbf{e} \in E(\Gamma')$. I.e., for every $i \in N$, $\mathbf{e}_{\hat{\imath}} \in \mathbf{K}_{\hat{\imath}}$ and $e_i \in R'_i(\mathbf{e}_{\hat{\imath}})$. As every R_i is proper, $R_i(\mathbf{e}_{\hat{\imath}}) \neq \emptyset$ ($i \in N$) holds. Now part 1 implies, as desired, $e_i \in R'_i(\mathbf{e}_{\hat{\imath}}) = R_i(\mathbf{e}_{\hat{\imath}})$ ($i \in N$). $\square$

5 Continuity of best-response correspondences

Proe:brcontinu

Proposition 5 *Consider a game in strategic form where each strategy set is a metric space. Let i be a player. Suppose the strategy set X_i is compact, the payoff function f_i is continuous and the best-response correspondence $R_i : \mathbf{X}_{\hat{\imath}} \multimap X_i$ is single-valued. Then the mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ is continuous.* $\diamond$

Proof.— A comfortable way to prove this, is with Berge's maximum theorem, i.e. with Theorem 5 in the appendix. Let us check its conditions (we apply it with "$F(\mathbf{z}) = X_i$)".

Well, X_i and $\mathbf{X}_{\hat{\imath}}$ are non-empty metric spaces, $f_i : X_i \times \mathbf{X}_{\hat{\imath}} \to \mathbb{R}$ is a continuous function and X_i is compact. For $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$, letting

$$m(\mathbf{z}) := \max_{x_i \in X_i} f_i^{(\mathbf{z})}(x_i),$$

we have $R_i(\mathbf{z}) = \{x_i \in X_i \mid f_i^{(\mathbf{z})}(x_i) = m(\mathbf{z})\}$. Theorem 5(3) guarantees that R_i is continuous. $\square$

Here is a proof "by hand" of this proposition: fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. We prove that R_i is continuous at $\mathbf{z}$. In order to do so we take a sequence $(\mathbf{z}_m)$ in $\mathbf{X}_{\hat{\imath}}$ with $\lim_{m\to\infty} \mathbf{z}_m = \mathbf{z}$ and show that, with $b_m := R_i(\mathbf{z}_m) \in X_i$, the sequence (b_m) is convergent with $\lim_{m\to\infty} b_m = R_i(\mathbf{z})$.

In order to do so we first prove that each convergent subsequence of (b_m) has limit $R_i(\mathbf{z})$. Well let $(b_{p(m)})$ be a convergent subsequence, say its limit is $b_\star \in X_i$. For any m, we have $f_i(b_{p(m)}; \mathbf{z}_{p(m)} \geq f_i(x_i; \mathbf{z}_{p(m)})$ $(x_i \in X_i)$. These inequalities together with the continuity of f_i imply $f_i(b_\star; \mathbf{z}) \geq f_i(x_i; \mathbf{z})$ $(x_i \in X_i)$. Thus $b_\star \in R_i(\mathbf{z})$. As $\#R_i(\mathbf{z}) = 1$, it follows, as desired, that $b_\star = R_i(\mathbf{z})$.

Further, as X_i is compact, the sequence (b_m) has a convergent subsequence $(b_{p(m)})$. So we have shown that each convergent subsequence of (b_m) has the same limit, i.e. $R_i(\mathbf{z})$. In other words: the sequence of (b_m) has $R_i(\mathbf{z})$ as unique limit point. This implies that the sequence (b_m) is convergent with $\lim_{m\to\infty} b_m = R_i(\mathbf{z})$.

Remembering that X_i is a best-response superset for player i, we see that the following proposition improves upon the previous one.

Pron:brcontinu2

Proposition 6 *Consider a game in strategic form where each strategy set is a metric space. Let i be a player. Suppose the payoff function f_i is continuous and the best-response correspondence $R_i : \mathbf{X}_{\hat{\imath}} \multimap X_i$ is single-valued. Then each of the following two conditions separately is sufficient for the mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ to be continuous.*

a. There exists a compact best-response superset for player i.

b. The strategy set X_i is a closed subset of a finite dimensional real normed linear space E and the best-response mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ is bounded. $\diamond$

Proof.— a. Let K_i be a compact best-response superset for player i. As the correspondence R_i is single-valued, K_i is not empty. As for every $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$, we have $R_i(\mathbf{z}) \neq \emptyset$, Proposition 1 implies for every $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ that

$$R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})} = \operatorname{argmax} f_i^{(\mathbf{z})} \restriction K_i.$$

This makes that again we can apply Berge's maximum theorem for obtaining the desired result: now apply it to the restriction of f_i to $K_i \times \mathbf{X}_{\hat{\imath}}$. In doing so it follows that mapping $R_i : \mathbf{X}_{\hat{\imath}} \to K_i$ is continuous. So also the mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ is continuous.

b. By Lemma 3(1), there exists a compact best-response superset. Now apply part a. $\square$

Remark: the two previous propositions where proved by referring to Berge's maximum theorem. Although this theorem supposes continuity of the function that is, for each parameter value, maximized, it does not suppose that for each parameter value there exists precisely one maximiser as we supposed in the two previous propositions.

The next propositon in turn improves upon the two above ones. It relies on a result that use the closedness of a graph.

Stee:aangepast

Theorem 1 *Consider a game in strategic form where every action set is a metric space. Fix $i \in N$. Suppose $f_i : \mathbf{X} \to \mathbb{R}$ is upper-semicontinuous and for every $x_i \in X_i$ the function $f_i^{(\cdot)}(x_i) : \mathbf{X}_{\hat{\imath}} \to \mathbb{R}$ is lower-semicontinuous and the best-response correspondence $R_i : \mathbf{X}_{\hat{\imath}} \multimap X_i$ is single-valued. Then each of the following two conditions separately is sufficient for the mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ to be continuous.*

a. There exists a compact best-response superset for player i.

b. The strategy set X_i is a closed subset of a finite dimensional real linear space E and the best-response mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ is bounded. $\diamond$

Proof.— a. Let K_i be a compact best-response superset for player i. As the correspondence R_i is single-valued, K_i is not empty. If we show that the mapping $R_i : \mathbf{X}_{\hat{\imath}} \to K_i$ is continuous, then also $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ is continuous. Further consider $R_i : \mathbf{X}_{\hat{\imath}} \to K_i$.

As K_i is a compact metric space and $\mathbf{X}_{\hat{\imath}}$ is a metric space, sufficient (and necessary) for R_i to be continuous is that R_i has a closed graph, i.e. that the set $\{(\mathbf{z}, R_i(\mathbf{z})) \mid \mathbf{z} \in \mathbf{X}_{\hat{\imath}}\}$ is closed in $\mathbf{X}_{\imath} \times K_i$. In order to show that this graph is closed, we take a convergent sequence $(\mathbf{z}^{(m)}, R_i(\mathbf{z}^{(m)}))$ from this graph with

$$\lim_{m \to \infty} (\mathbf{z}^{(m)}, R_i(\mathbf{z}^{(m)})) = (\mathbf{z}, k_i) \in \mathbf{X}_{\imath} \times K_i$$

and show that $k_i = R_i(\mathbf{z})$, i.e. that $f_i(k_i; \mathbf{z}) \geq f_i(y; \mathbf{z})$ $(y \in K_i)$. So fix $y \in K_i$. Note that $\lim_{m \to \infty} \mathbf{z}^{(m)} = \mathbf{z}$, $\lim_{m \to \infty} R_i(\mathbf{z}^{(m)}) = k_i$ and $f_i(R_i(\mathbf{z}^{(m)}); \mathbf{z}^{(m)}) \geq f_i(y; \mathbf{z}^{(m)})$. As f_i is upper-semicontinuous in $(k_i; \mathbf{z})$ and $f_i^{(\cdot)}(y)$ is lower-semicontinuous in $\mathbf{z}$, we obtain, as desired,

$$f_i(k_i; \mathbf{z}) \geq \limsup_{m \to \infty} f_i(R_i(\mathbf{z}^{(m)}); \mathbf{z}^{(m)}) \geq$$

$$\limsup_{m \to \infty} f_i(y; \mathbf{z}^{(m)}) \geq \liminf_{m \to \infty} f_i(y; \mathbf{z}^{(m)}) = \liminf_{m \to \infty} f_i^{(\mathbf{z}^{(m)})}(y)$$

$$\geq f_i^{(\mathbf{z})}(y) = f_i(y; \mathbf{z}).$$

b. By Lemma 3(1), there exists a compact best-response superset. Now apply part a. $\square$

6 Nash Equilibrium existence via Brouwer's fixed point theorem

Proe:nashbr

Proposition 7 *Consider a game in strategic form where each strategy sets X_i is a closed convex subset of a finite dimensional normed real linear space and where best-response correspondences $R_i : \mathbf{X}_{\hat{\imath}} \multimap X_i$ are single-valued. If each best-response mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ is continuous, then each of the following two conditions is sufficient for the game to have a Nash equilibrium.*

a. Each strategy set is compact.

b. Each player has a bounded domination set. $\diamond$

Proof.— Suppose each mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$ is continuous.

a. Also the joint best-response mapping $\mathbf{R} : \mathbf{X} \to \mathbf{X}$ is continuous. As each X_i is a non-empty convex compact subset of, say, E_i, it follows that $\mathbf{X}$ is a non-empty convex compact subset of $E := E_1 \times \cdots \times E_n$. As E is finite dimensional, Brouwer's fixed point theorem guarantees that $\mathbf{R}$ has a fixed point. Thus the game has a Nash equilibrium.

2. By Lemma 3(2b), each player i has a convex compact domination set K_i. Note that, by Proposition 2, K_i also is a best-response superset for player i. Using the notations Γ and Γ' of Proposition 4, the game Γ' is well-defined.

With the R'_i the best-response correspondences in Γ', Proposition 4(1) implies $R_i(\mathbf{z}) = R'_i(\mathbf{z})$ ($\mathbf{z} \in \mathbf{K}_{\hat{\imath}}$). Thus the R'_i are single-valued. As the R_i are continuous, the R'_i are continuous too.

Part a applies to Γ' and guarantees that $E(\Gamma') \neq \emptyset$. Now Proposition 4(3) guarantees that $E(\Gamma) = E(\Gamma')$. Thus $E(\Gamma) = E(\Gamma') \neq \emptyset$. □

`Stee:brouwerbounded`

Theorem 2 *Consider a game in strategic form where each strategy set is a convex closed subset of a finite dimensional normed real linear space. Suppose payoff functions are continuous and conditional payoff functions are strictly quasi-concave. Then each of the following two conditions is sufficient for the game to have a Nash equilibrium.*

a. Each strategy set is compact.

b. Each player has a bounded domination set. ⋄

Proof.— It is sufficient to prove part b. By Lemma 4, each best-response correspondence R_i is single-valued and thus a mapping $R_i : \mathbf{X}_{\hat{\imath}} \to X_i$. Proposition 5 guarantees that these mappings are continuous. Proposition 7(a) guarantees that the game has a Nash equilibrium. □

7 Generalised Nikaido-Isoda Theorem

The following powerful Nash equilibrium existence result (cfr. with Theorem 2) being a variant of a result in [6] by Nikaido and Isoda, holds:

`Stee:nikaidoisoda`

Theorem 3 *A game in strategic form where the strategy sets are convex compact subsets of a finite dimensional normed real linear space, payoff functions are continuous and conditional payoff functions are quasiconcave has a Nash equilibrium.* ⋄

Proof.— See [4] (en [3]). The proof there refers to Kakutani's fixed point theorem. □

Note that (because of the Weierstrass' theorem) best-response correspondences in this theorem are proper, but not necessarily single-valued.

Remark: the reason that the author refers to Theorem 3 as a variant of a result in [6] by Nikaido and Isoda, is that various other authors do this; see for example [4, 3, 5]. However, the article [2] already essentially contains this result.

Below we shall with Theorem 4 improve upon Theorem 3 concerning the compactness of strategy sets.

`Stee:improvement`

Theorem 4 *(Generalized Nikaido-Isoda theorem.) Consider a game in strategic form with strategy sets that are closed convex subsets of a finite dimensional normed real linear space, continuous payoff functions and quasi-concave conditional payoff functions. Then each of the following two conditions separately is sufficient for the game to have a Nash equilibrium.*

1. *Each player has a bounded domination set.*

2. *Each strategy set is compact.* ⋄

Proof.— 1. By Lemma 3(2b) each player has a convex compact domination set, say K_i. Note that, by Proposition 2, K_i also is a best-response superset for player i. Using the notations Γ and Γ' of Proposition 4, the game Γ' is well-defined. Theorem 3 applies to Γ' and guarantees that $E(\Gamma') \neq \emptyset$. By Lemma 4(1), each best-response correspondence R_i is proper. Proposition 4(3) guarantees that $E(\Gamma) = E(\Gamma')$. Thus $E(\Gamma) = E(\Gamma') \neq \emptyset$.

2. As a compact strategy set is a bounded domination set. □

A Berge's maximum theorem

The following theorem concerns Berge's maximum theorem; e.g. see [1].

Stee:berge

Theorem 5 *Let X and T be non-empty metric spaces, $F : T \multimap X$ a compact-valued proper correspondence and $f : X \times T \to \mathbb{R}$ a continuous function. Let*[3] *the function $m : T \to \mathbb{R}$ be well-defined by*

$$m(t) := \max_{x \in F(t)} f(x, t)$$

and let the compact-valued proper correspondence $M : T \multimap X$ be well-defined by

$$M(t) := \{x \in F(t) \mid f(x, t) = m(t)\}.$$

1. *Let $t \in T$. If F is continuous at t, then M is upper hemicontinuous at t and m is continuous at t.*

2. *If F is continuous, then M is hemicontinuous and m is continuous.*

3. *If F is continuous and M is single-valued, then M interpreted as a mapping $T \to X$ is continuous.* $\diamond$

B Brouwer's fixed point theorem

Theorem 6 *(Brouwer's fixed point theorem) Let D be a non-empty convex compact convex set in a finite dimensional normed real linear space and $F : D \to D$ a continuous mapping. Then F has (at least) one fixed point x.*

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[3] The well-definedness of m follows from the Weierstrass theorem and that of M concerns a general topological result about compactness of the set of maximisers of a continuous function with domain a compact metric space.