Game Theory

Lesson 6: Nim

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What You will learn

After studying Lesson 6, You

- Determine the value of a given Nim game.
- Being able to play according to an optimal strategy.

The rules of Nim

Nim already was introduced in Lesson 1. We repeat its rules here again.

Nim is the following two player game. A certain number of piles consisting of a certain number of matches is put together. Both players take turns. Player 1 starts. Each turn a player must remove at least one match from a pile. The player that takes the last matche(s) wins. So there are infinitely many Nim games possible.

We considered Nim $(2, 3, 5, 5, 3, 2)$, i.e. there is one pillow with 2 matches, one with 3, ..., and one with 2. And now know that player 2 can win this game by “imitating” the moves of player 1. I.e. if player 1 takes away 2 matches in pillow 3, then player 2 takes away two matches in pillow 4.

History

In Wikipedia we read: variants of Nim have been played since ancient times. The game is said to have originated in China. Its current name was coined by Charles L. Bouton of Harvard University, who also developed the complete theory of the game in 1901. The Oxford English Dictionary derives the name from the German verb nimm, meaning "take".

Positional numeral systems

The natural numbers can be represented in many ways. Such a way is called numeral system. It uses digits or other symbols in a consistent manner.

Very interesting are positional numeral systems. They concern the extension to any base of the Hindu-Arabic numeral system (base ten, or decimal system). Base-ten is the most common for most of us. Besides base-ten, important systems are base-two and base-sixteen

In a positional numeral system the contribution of a digit to the value of a number is the value of the digit multiplied by a factor determined by the position of the digit. It works by using exponentiation of the base.

Positional numeral systems (ctd.)

Base-ten: the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are used. Example: 304 represents the number $3x10^2 + 0x10^1 + 4x10^0 = 304$.

Base-two: the digits $0, 1$ are used. Example: 110 represents the number $1x2^2 + 1x2^1 + 0x2^0 = 6$.

Base-16: the digits $0, 1, 2, 3, 4, 5, 5, 7, 8, 9, A = 10, B = 11, C = 12, D = 13, E = 14, F = 15$ are used. Example: $14B9$ represents the number $1x16^3 + 4x16^2 + 11x16^1 + 9x16^0 = 5305$.

For analysing Nim, base-two (binary numbers) is very useful, as we shall see.

Binary numbers

Base – ten	*Base – two digits*
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110

From decimal to binary

To convert numbers from decimal to binary, the given decimal number is divided repeatedly by 2 and the remainders are noted down till we get 0 as the final quotient.

The following steps provide a procedure of conversion.

Step 1: Divide the given decimal number by 2; note the remainder.
Step 2: Divide the obtained result by 2; note the remainder.
Step 3: Repeat the above steps until you get 0 as the result.
Step 4: Now, write the remainders in such a way that the last remainder is written first, followed by the rest in the reverse order.

From decimal to binary (ctd.)

Here is an example where we convert the decimal number 20 into a binary one.

$$\begin{aligned} 20 : 2 &= 10 \text{ remainder } 0 \\ 10 : 2 &= 5 \text{ remainder } 0 \\ 5 : 2 &= 2 \text{ remainder } 1 \\ 2 : 2 &= 1 \text{ remainder } 0 \\ 1 : 2 &= 0 \text{ remainder } 1 \end{aligned}$$

Result: 10100

Optimal strategy: Nim-sum

Given a position of a Nim game.
Determine in base-two for each pillow the number of matches.
Write these numbers down, one below the other.
Add in base-ten the column sums.
The result is a sequence of numbers, referred to as the Nim-sum .

A Nim-sum is called even if all of its entries are even numbers and odd otherwise.

Optimal strategy: Nim-sum (ctd.)

Example: Consider Nim $(5, 7, 6, 4, 1, 3, 9)$.

0101

0111

0110

0100

0001

0011

1001

$------$

1435

The Nim-sum is $1, 4, 3, 5$ which is an odd one.

Optimal strategy

The following provides an optimal strategy for each of the players.

Consider a (non-final) position of a Nim-game. Determine the Nim-sum.

If this sum is even, then the player who has to move cannot guarantee himself a win.

If this sum is odd, than the player who has to move can guarantee himself a win by playing an optimal move. An optimal move is to create a position with an even Nim-sum.

Proof: see Exercise set 5.

Combinatorial games

Nim is a very fundamental game. Loosely speaking, the reason is that combinatorial games can be analysed by splitting them into Nim games. This concerns the so-called Sprague-Grundy theorem.

Our hero: John Nash

- John Nash (1928 – 2015). 
- Mathematician.
- Nobel Price for economics in 1994, together with Harsanyi and Selten.
- Got this price for his PhD dissertation (27 pages) in 1950. Used a theorem of Brouwer.
- Abel Price for mathematics in 2015. Just after receiving it he was killed in a car crash.
- Enjoy looking to the following video: `http://topdocumentaryfilms.com/a-brillant-madness-john-nash`

Luitzen-Jan Egbertus Brouwer

- Luitzen Jan Egbertus Brouwer (1881-1966).
- Dutch mathematician, Frisian and idealist.
- Brouwer proved a number of theorems that were breakthroughs in the emerging field of topology. Most famous is his fixed point theorem.
- He died after he was strucked by a vehicle while crossing the street in front of his house.