Game Theory

Lesson 3: Games in Strategic Form

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2024, Period 4

What You will learn

After studying Lesson 3, You

- should be familiar with the notion of game in strategic form;
- should understand for games in strategic form, the introduced game theoretic vocabulary formed by the fundamental notions;
- should know how to apply the things You have learned to concrete examples;
- should know the fundamental result for antagonistic games.

In addition:

- You should be familiar with the specific game, i.e. Cournot Duopoly, introduced in this lesson.

Lesson 2 was devoted to bimatrix games. In the context of these games You learned there various game theoretical notions. The most important one being the notion of Nash equilibrium which is used to make predictions how games will be played.

Bimatrix games concern games where two players simultaneously and independently choose a strategy and each player has a finite number of strategies. Concerning our concrete games (i.e. Tic-Tac-Toe, Hex, Cournot Duopoly (see below), Hotelling Game, Nim), only the Hotelling Game is a bimatrix game. The Cournot Duopoly is not as each player there has infinite many strategies.

Below I first shall now show how the Hotelling Game can be represented as a bimatrix game. Next the notion of game in strategic form is introduced, which allows us to handle the games with more than two players.

Hotelling Game

Please, if needed, review Lesson 2 for the definition of the Hotelling Game. Now consider this game of three sites: $0, 1$ and 2. (So $m = 2$)
This game can be represented as a 3×3-bi-matrix game with, for player 1 at the first row strategy 0, at the second row strategy 1, at the third row strategy 2. And with the same convention for player 2.

If You do this correctly (and do it!), then You find

$$\begin{pmatrix} 3/2; 3/2 & 1; 2 & 3/2; 3/2 \\ 2; 1 & 3/2; 3/2 & 2; 1 \\ 3/2; 3/2 & 1; 2 & 3/2; 3/2 \end{pmatrix}.$$

Hotelling Game (ctd.)

The game has a unique Nash equilibrium: the strategy profile $(1, 1)$. So the players locate in the middle.

In Exercise 5 of Exercises 2 You will show that this result does not depend on the m which determines the number of sites.

Game in strategic form

Here is the announced generalisation of a bimatrix game.

Definition

Game in strategic form , specified by

- n players : $1, \ldots, n$.
- for each player i a strategy set X_i.
- for each player i a payoff function $f_i(x_1, \ldots, x_n)$.

An element x_i of X_i is called strategy . And a strategy profile is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with the $x_i \in X_i$.

Interpretation is the same as that for a bimatrix game. In particular: players choose simultaneously and independently a strategy.

Game in strategic form (ctd.)

A game in strategic form is called finite if each strategy set X_i is finite.

Of course, in the case of two players a finite game in strategic form can be represented as a bimatrix game. For example if each player has two strategies, say $X_1 = X_2 = \{1, 2\}$, then the bimatrix game is

$$\begin{pmatrix} f_1(1,1); f_2(1,1) & f_1(1,2); f_2(1,2) \\ f_1(2,1); f_2(2,1) & f_1(2,2); f_2(2,2) \end{pmatrix}.$$

Please note that a game in strategic form is a game with imperfect information as the moves are simultaneously.

Fundamental notions

The fundamental notions for bimatrix games in Lesson 1 (strictly dominant strategy, strictly dominant equilibrium, Nash equilibrium, weakly Pareto efficient strategy profile, fully cooperative strategy profile, prisoner's dilemma) also make sense for an arbitrary game in strategic form. Their definition is exactly the same. (The notion of zero-sum game however presupposes two players.)

It may be a good idea to review these notions.

Fundamental notions (ctd.)

In addition to the above fundamental notions, we introduce the following

- Best reply correspondence R_i of player i: set of best strategies of the player given the strategies of his opponents.

In case R_i is a function, i.e. there is always one unique best strategy, one also speaks of reaction function.

The following fundamental relation between Nash equilibria and the best reply correspondences hold for the case of two players: a strategy profile (x_1, x_2) is a Nash equilibrium if and only if

$$x_1 \in R_1(x_2) \text{ and } x_2 \in R_2(x_1),$$

i.e. if no player regrets his choice.
(Analogous result holds for more than 2 players).

Example

Determine the best reply correspondence and the Nash equilibria of the game

$$\begin{pmatrix} 2;4 & 1;4 & 4;3 & 3;0 \\ 1;1 & 1;2 & 5;2 & 6;1 \\ 1;2 & 0;5 & 3;4 & 7;3 \\ 0;6 & 0;4 & 3;4 & 1;5 \end{pmatrix}.$$

Answer: $R_1(1) = \{1\}, R_1(2) = \{1,2\}, R_1(3) = \{2\}, R_1(4) = \{3\}, R_2(1) = \{1,2\}, R_2(2) = \{2,3\}, R_2(3) = \{2\}, R_2(4) = \{1\}$.
Nash equilibria: $(1,1), (1,2), (2,2)$ and $(2,3)$.

Cournot Duopoly

A special topic in economics is industrial organization. The modern theory of industrial organization heavily relies on game theory; various market forms are considered. Here we consider the market form of Cournot Oligopoly . The Cournot Oligopoly is one of the oldest economic games.

A Cournot Oligopoly concerns firms in a competitive setting. There are various variants. For us the homogeneous Cournot Duopoly is the most important: ‘duopoly’ concerns the assumption of two firms and ‘homogeneous’ that the firms sell the same article.

Cournot Duopoly (ctd.)

In terms of our vocabulary: a Cournot Oligopoly. is a game in strategic form with the firms as players and a real interval containing 0, like $[0, +\infty[$, as common strategy set and the profit functions as payoff functions.

The model is as follows: n firms simultaneously and independently supply an amount of the article to the market and then can sell it for a price depending on the total amount. With x_i the amount for firm i, the total amount is $X = x_1 + \cdots + x_n$ and the price is $p(X)$. The function p is called price function (or, 'par abus de langage', inverse demand function). With c_i the cost function of firm i the profit of firm i, being revenue minus costs, is

$$\pi_i(x_1, \ldots, x_n) = p(X)x_i - c_i(x_i).$$

The function π_i is called profit function of firm i.

A Nash equilibrium of a Cournot Oligopoly is called Cournot

Nash equilibria in the continuous case

So in this lesson we also deal with games, like the Cournot Duopoly, where each player has infinitely many strategies. For such games one needs calculus in order to determine various fundamental objects, like Nash equilibria and fully cooperative strategy profiles.

How to do this? I'll explain it in the case of two players with payoff functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$.

Nash equilibria in the continuous case (ctd.)

Remember that in a Nash equilibrium no player regrets his choice. More formally: a strategy profile (e_1, e_2) is a Nash equilibrium if e_1 maximises $f_1(x_1, e_2)$ as a function of x_1 and e_2 maximises $f_2(e_1, x_2)$ as a function of x_2.

Under conditions where one can find maxima by putting derivatives to zero (in economics such conditions are quite usual) one can find the Nash equilibria by solving the two equations

$$\frac{\partial f_1}{\partial x_1} = 0 \text{ and } \frac{\partial f_2}{\partial x_2} = 0$$

in the two unknowns x_1, x_2.

Fully cooperative strategy profile in the continuous case

Now i explain how to find fully cooperative strategy profiles in the continuous case.

Well, a strategy profile is fully cooperative if it maximises the total payoff function $f(x_1, x_2) = f_1(x_1, x_2) + f_2(x_1, x_2)$. This is a function of two variables.

From mathematics we know that in order to maximise this function one has (again under suitable assumptions) to put equal to zero both partial derivatives:

$$\frac{\partial f}{\partial x_1} = 0 \text{ and } \frac{\partial f}{\partial x_2} = 0.$$

Example

Consider the following Cournot Duopoly. Assume assume a price function

$$p(X) = 60 - X,$$

and cost functions

$$c_1(x_1) = x_1^2, \;\; c_2(x_2) = 15x_2 + x_2^2.$$

The profit functions are

$$\pi_1(x_1, x_2) = (60 - (x_1 + x_2))x_1 - x_1^2 = 60x_1 - 2x_1^2 - x_1 x_2,$$

$$\pi_2(x_1, x_2) = (60 - (x_1 + x_2))x_2 - (15x_2 + x_2^2) = 60x_2 - 2x_2^2 - x_1 x_2 - 15x_2.$$

Example (ctd.)

We have

$$\frac{\partial \pi_1}{\partial x_1} = 60 - 4x_1 - x_2,$$
$$\frac{\partial \pi_2}{\partial x_2} = 60 - 4x_2 - x_1 - 15.$$

Solving

$$\frac{\partial \pi_1}{\partial x_1} = 0 \text{ and } \frac{\partial \pi_2}{\partial x_2} = 0,$$

one finds (please do it!) $x_1 = 13, x_2 = 8$. Thus the strategy profile

$$(13, 8).$$

is a unique Nash equilibrium.

Let us calculate the profits in this equilibrium. One finds

$$\pi_1(13, 8) = 338 \text{ and } \pi_2(13, 8) = 128.$$

So the total profit is 466.

Example (ctd.)

Now consider the case where the two firms collude, i.e. from a cartel, with goal to maximise joint profit.

The joint profit function is

$$\pi(x_1, x_2) = 60x_1 + 60x_2 - 2x_1^2 - 2x_2^2 - 2x_1x_2 - 15x_2.$$

Solving $\frac{\partial \pi}{\partial x_1} = 0$ and $\frac{\partial \pi}{\partial x_2} = 0$ gives $x_1 = 67/6$ and $x_2 = 23/3$. Thus $(67/6, 23/3)$ is a unique fully cooperative strategy profile. And we find

$$\pi(67/6, 23/3) = 2861/6 = 476.8..(> 466).$$

Antagonistic game: fundamental result

Theorem

Consider an antagonistic game. If (a_1, a_2) and (b_1, b_2) are Nash equilibria, of an antagonistic game then $f_1(a_1, a_2) = f_1(b_1, b_2)$ and $f_2(a_1, a_2) = f_2(b_1, b_2)$.

Proof.

$f_1(a_1, a_2) \geq f_1(b_1, a_2) = -f_2(b_1, a_2) \geq -f_2(b_1, b_2) = f_1(b_1, b_2)$. In the same way $f_1(b_1, b_2) \geq f_1(a_1, a_2)$. Therefore $f_1(a_1, a_2) = f_1(b_1, b_2)$ and thus also $f_2(a_1, a_2) = f_2(b_1, b_2)$. □

(Please try to understand each step in the above proof!) This is a very important result, as we shall see in Lesson 4.