Game Theory

Lesson 5: Congestion

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What You will learn

After studying Lesson 5, You

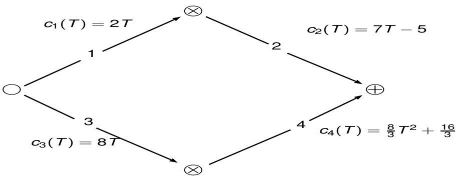
- should be able to perform a game theoretic analysis of simple congestion games.

Introduction

In this lesson we are going to consider the real world problem of congestion and present a simple game theoretic model for it. In fact, below You can find a quick and efficient route for understanding the very basics of congestion games.

Let us start with a very simple example by considering the following traffic network:

Simple traffic network



The intended interpretation is as follows.

- Each morning *n* commuters want to go from node (i.e. place) $\bigcirc$ to node $\bigoplus$.
- There are 4 roads: 1, 2, 3, 4. The configuration of these roads makes that there are two routes for commuting: roads 1–2 (route 1) and roads 3–4 (route 2).
- $c_j(T)$ denotes the costs for a commuter of using road j if T commuters use this road. (So this costs are the same for all commuters who take the road.)

Questions

Questions we want to answer:

- How the commuters will behave?
- Is this behaviour social optimal?
- Is it Pareto efficient?

We shall answer these questions by looking to them from a game theoretical perspective. In order to do so we make out of situations as the above one (with only two routes) as follows a game in strategic form.

Of course we assume that the commuters are rational and intelligent. But also that they simultaneously and independently choose a route.

Game structure

The commuters are the players and the strategy set of a commuter is the set of routes he can use. Note that in the above simple model each commuter has the same strategy set. We label (in some way) the commuters by $1, 2, \ldots, n$ and the strategies by $1, 2, 3, \ldots$.

Denote by $(x_1, \ldots, x_n)$ a strategy profile, i.e. player 1 plays x_1, player 2 plays x_2, ...

Analysis

First let us suppose $n = 2$, i.e. that there are 2 commuters. Denote by $C_1(x_1, x_2)$ the total costs of commuter 1 if this commuter chooses strategy x_1 and commuter 2 strategy x_2. Define $C_2(x_1, x_2)$ is the same way.

For example: at the strategy profile $(2, 1)$ (i.e. player 1 takes route 2 and player 2 takes route 2), player 1 has costs $8 \cdot 1$ for road 3 and $\frac{8}{3}1^2 + \frac{16}{3} = 8$ for road 4, thus $C_1(2, 1) = 8 + 8 = 16$. And for player 2 this leads to $C_2(2, 1) = 2 + 2 = 4$.

Analysis (ctd)

We find
$C_1(1,1) = 13,\ C_2(1,1) = 13$
$C_1(1,2) = 4,\ C_2(1,2) = 16$
$C_1(2,1) = 16,\ C_2(2,1) = 4$
$C_1(2,2) = 32,\ C_2(2,2) = 32$

This can be represented as follows by means of a so-called bimatrix:

$$\begin{pmatrix} 13;13 & 4;16 \\ 16;4 & 32;32 \end{pmatrix}.$$

Analysis (ctd)

$$\begin{pmatrix} 13; 13 & 4; 16 \\ 16; 4 & 32; 32 \end{pmatrix}.$$

A simple game theoretic analysis shows the following.

Prediction of behaviour : both choose route 1.

Social optimal : each commuter chooses a different route.

We see:

Equilibrium is not social optimal. However, equilibrium (for our case) is Pareto efficient.
The case of more than two commuters is more difficult to handle.

Fundamental result

There are various results about congestion games, like the existence of Nash equilibria. The first one was:

[redacted]

Rosenthal; A Class of Games Possessing Pure-Strategy Equilibria; International Journal of Game Theory; 1973.

Pioneers concerning congestion games: Rosenthal, Milchtaich, Monderer, and Shapley (Nobel Prize Economics).

Braess' Paradox

The Braess' Paradox is named after the mathematician Dietrich Braess. It states that adding a link to a transportation network can increase the travel cost for all commuters in the network. It is a counterintuitive phenomenon.

The paradox occurs only in networks in which the commuters operate independently and noncooperatively, in a decentralized manner.

In fact the Braess' Paradox also occurs in other types of networks. This makes this paradox extra interesting!

Braess' Paradox (ctd.)

Let us now look to the following Youtube video:
`https://www.youtube.com/watch?v=cALezV_Fwi0`

Not only in Seoul as mentioned in this video, but also in other cities as New York and Stuttgart the Braess' paradox has been observed.

John Nash

Our hero:

- John Nash (1928 – 2015).
- Mathematician.
- Nobel price for economics in 1994, together with Harsanyi and Selten.
- Abel Price for mathematics in 2015. Just after having received it he was killed in a car crash.
- Got this price for his PhD dissertation (27 pages) in 1950.
- Enjoy looking to the following video about our main hero:

  http://topdocumentaryfilms.com/a-brilliant-madness-john-nash.