# Game Theory

P. v. Mouche

Exercises 2

**Exercise 1** *Determine which of the following bi-matrix games are a prisoner's dilemma.*

*a.* $\begin{pmatrix} 3;-1 & 3;1 & 6;0 \\ 1;0 & 3;1 & 6;0 \\ 2;2 & 4;1 & 8;2 \end{pmatrix}.$

*b.* $\begin{pmatrix} 1;0 & 3;1 & 6;0 \\ 2;1 & 4;1 & 8;1 \end{pmatrix}.$

*c.* $\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}.$

*d.* $\begin{pmatrix} -1;-1 & 2;0 \\ 0;2 & 3;3 \end{pmatrix}.$

*e.* $\begin{pmatrix} 2;2 & -1;3 \\ 3;-1 & 0;0 \end{pmatrix}.$

**Exercise 2** *The following true/false statements concern an arbitrary bi-matrix game.*

a. *This concerns a game with two players.*

b. *The game has at least one Nash equilibrium.*

c. *The game has a strictly dominant strategy.*

d. *The game has a fully cooperative strategy profile.*

e. *The has a weakly Pareto efficient strategy profile.*

f. *Each fully cooperative strategy profile is weakly Pareto efficient.*

g. *A strictly dominant strategy is fully cooperative.*

h. *If the game is a prisoners' dilemma, then it has a Nash equilibrium.*

i. *It is impossible that a weakly Pareto inefficient strategy profile is a Nash equilibrium.*

**Exercise 3** *The following true/false statements deal with the bi-matrix game*

$$\begin{pmatrix} 3;6 & 6;5 & 7;-3 \\ -6;2 & 5;3 & 5;4 \end{pmatrix}.$$

a. *The row-player has 2 strategies.*

*b. There are* 6 *strategy profiles.*

*c. The strategy profile* $(1, 1)$ *is a Nash equilibrium.*

*d. The row-player has a strictly dominant strategy.*

*e. There is a weakly Pareto inefficient Nash equilibrium.*

*f. The column-player has a strictly dominant strategy.*

*g. This game is a prisoners' dilemma.*

*h. Playing row* 1 *and column* 3 *is a fully cooperative strategy profile*

*i. This game is a zero-sum game.*

*j.* $(1, 2)$ *is a weakly Pareto efficient strategy profile.*

**Exercise 4** *A new notion: a strict Nash equilibrium is a Nash equilibrium with the property that if a player deviates from his strategy in this Nash equilibrium, his payoff will become less.*

*Given the following bimatrix game:*

$$\begin{pmatrix} 3;8 & -4;8 & 2;3 \\ 1;7 & 2;6 & 8;1 \\ 3;3 & 4;4 & 2;2 \\ 1;1 & 1;-1 & 1;-1 \end{pmatrix}.$$

*a. Determine the best reply correspondences.*

*b. Determine the strictly dominant strategies.*

*c. Determine the Nash equilibria.*

*d. Determine the strict Nash equilibria.*

*e. Determine the weakly Pareto-efficient strategy profiles.*

**Exercise 5** *Consider the Hotelling Game with sites* $0, 1, \ldots, m$ *where* $m$ *is even.*

*a. Show that for the payoff function* $f_1$ *of player* 1

$$f_1(x_1, x_2) := \begin{cases} \frac{x_1+x_2+1}{2} \textit{ if } x_1 < x_2, \\ \frac{m+1}{2} \textit{ if } x_1 = x_2, \\ m + 1 - \frac{x_1+x_2+1}{2} \textit{ if } x_1 > x_2 \end{cases}$$

*b. Show that* $(m/2, m/2)$ *is a Nash equilibrium.*

Short solutions.

*Solution 1* Only the game in e.

*Solution 2* aT bF cF dT eT fT gF hT iF.

*Solution 3* aT bT cT dT eF fF gF hF iF jT.

*Solution 4* a. $R_1(1) = \{1,3\}$, $R_1(2) = \{1,3\}$, $R_1(3) = \{2\}$, $R_2(1) = \{1,2\}$, $R_2(2) = \{1\}$, $R_2(3) = \{1,2\}$, $R_2(4) = \{1\}$.

a.

Strictly dominant strategies: do not exist.

b. They are $(1,1)$ (i.e. row 1 and column 1) and $(3,2)$.

c. $(3,2)$.

e. $(1,1),(1,2)(2,3),(3,2)$.

*Solution 5* a. Make a figure and count the contributions.

b. We have to show that $f_1(x_1, m/2) \leq f_1(m/2, m/2)$ for all $x_1$ and that $f_2(m/2, x_2) \leq f_2(m/2, m/2)$ for all $x_2$. We prove here the first statement; the second follows in the same way.

For $x_1 = m/2$, the statement is clear. For $x_1 < m/2$, we have, using part a, $f_1(x_1, m/2) = \frac{x_1 + \frac{m}{2} + 1}{2} = \frac{x_1}{2} + \frac{m}{4} + \frac{1}{2} < \frac{m}{4} + \frac{m}{4} + \frac{1}{2} = \frac{m+1}{2} = f_1(\frac{m}{2}, \frac{m}{2})$. And for $x_1 > m/2$, we have, using part a, $f_1(x_1, m/2) = m + 1 - \frac{x_1 + \frac{m}{2} + 1}{2} = m + 1 - \frac{x_1}{2} - \frac{1}{2} - \frac{m}{4} = \frac{3}{4}m + \frac{1}{2} - \frac{x_1}{2} > \frac{3}{4}m + \frac{1}{2} - \frac{m}{4} = \frac{m+1}{2} = f_1(\frac{m}{2}, \frac{m}{2})$.