Math Review

Dušan Drabik

de Leeuwenborch 2105

Dusan.Drabik@wur.nl

The material contained in these slides draws heavily on:
Geoffrey A. Jehle and Philip J. Reny (2011). Advanced Microeconomic Theory (3rd Edition). Prentice Hall, 672 p.

Basic definitions

A **set** is any collection of elements. Sets can be defined by *enumeration* of their elements, e.g., $S = \{2, 4, 6, 8\}$, or by *description* of their elements, e.g., $S = \{x \mid x \text{ is a positive even integer greater than zero and less than } 10\}$. When we wish to denote membership or inclusion in a set, we use the symbol $\in$. For example, if $S = \{2, 5, 7\}$, we say that $5 \in S$.

A set S is a **subset** of another set T if every element of S is also an element of T. We write $S \subset T$ (S is contained in T) or $T \supset S$ (T contains S). If $S \subset T$, then $x \in S \Rightarrow x \in T$.

Two sets are **equal sets** if they each contain exactly the same elements. We write $S = T$ whenever $x \in S \Rightarrow x \in T$ and $x \in T \Rightarrow x \in S$. Thus, S and T are equal sets if and only if $S \subset T$ *and* $T \subset S$. For example, if $S = \{\text{integers}, x \mid x^2 = 1\}$ and $T = \{-1, 1\}$, then $S = T$.

A set S is **empty** or is an **empty set** if it contains no elements at all. For example, if $A = \{x \mid x^2 = 0 \text{ and } x > 1\}$, then A is empty. We denote the empty set by the symbol $\emptyset$ and write $A = \emptyset$.

n-space is defined as the set product

$$\mathbb{R}^n \equiv \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \equiv \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 1, \ldots, n\}.$$

Convex Sets

Convex Sets in $\mathbb{R}^n$

$S \subset \mathbb{R}^n$ is a convex set if for all $\mathbf{x}^1 \in S$ and $\mathbf{x}^2 \in S$, we have

$$t\mathbf{x}^1 + (1 - t)\mathbf{x}^2 \in S$$

for all t in the interval $0 \leq t \leq 1$.

The Intersection of Convex Sets is Convex

Let S and T be convex sets in $\mathbb{R}^n$. Then $S \cap T$ is a convex set.

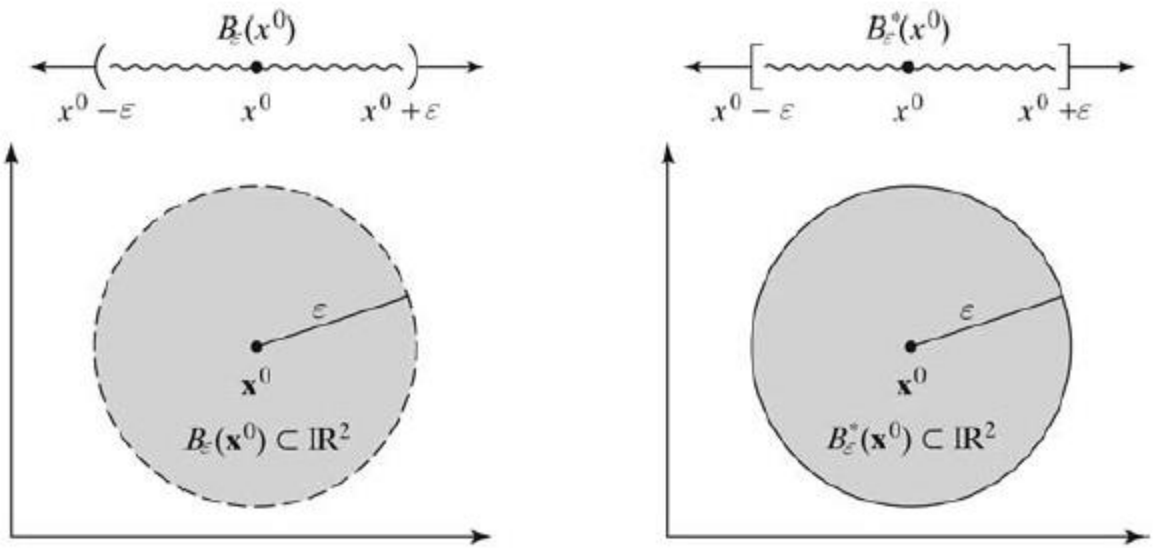
Open and Closed ε-Balls

1. *The open* ε*-ball with centre* $\mathbf{x}^0$ *and radius* $\varepsilon > 0$ *(a real number) is the subset of points in* $\mathbb{R}^n$*:*

$$B_\varepsilon(\mathbf{x}^0) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \underbrace{d(\mathbf{x}^0, \mathbf{x}) < \varepsilon}_{\textit{strictly less than}} \}$$

2. *The closed* ε*-ball with centre* $\mathbf{x}^0$ *and radius* $\varepsilon > 0$ *is the subset of points in* $\mathbb{R}^n$*:*

$$B^*_\varepsilon(\mathbf{x}^0) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \underbrace{d(\mathbf{x}^0, \mathbf{x}) \leq \varepsilon}_{\textit{less than or equal to}} \}$$



Open Sets in $\mathbb{R}^n$

$S \subset \mathbb{R}^n$ is an open set if, for all $\mathbf{x} \in S$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset S$.

On Open Sets in $\mathbb{R}^n$

1. *The empty set, $\emptyset$, is an open set.*
2. *The entire space, $\mathbb{R}^n$, is an open set.*
3. *The union of open sets is an open set.*
4. *The intersection of any finite number of open sets is an open set.*

Closed Sets in $\mathbb{R}^n$

S is a closed set if its complement, S^c, is an open set.

On Closed Sets in $\mathbb{R}^n$

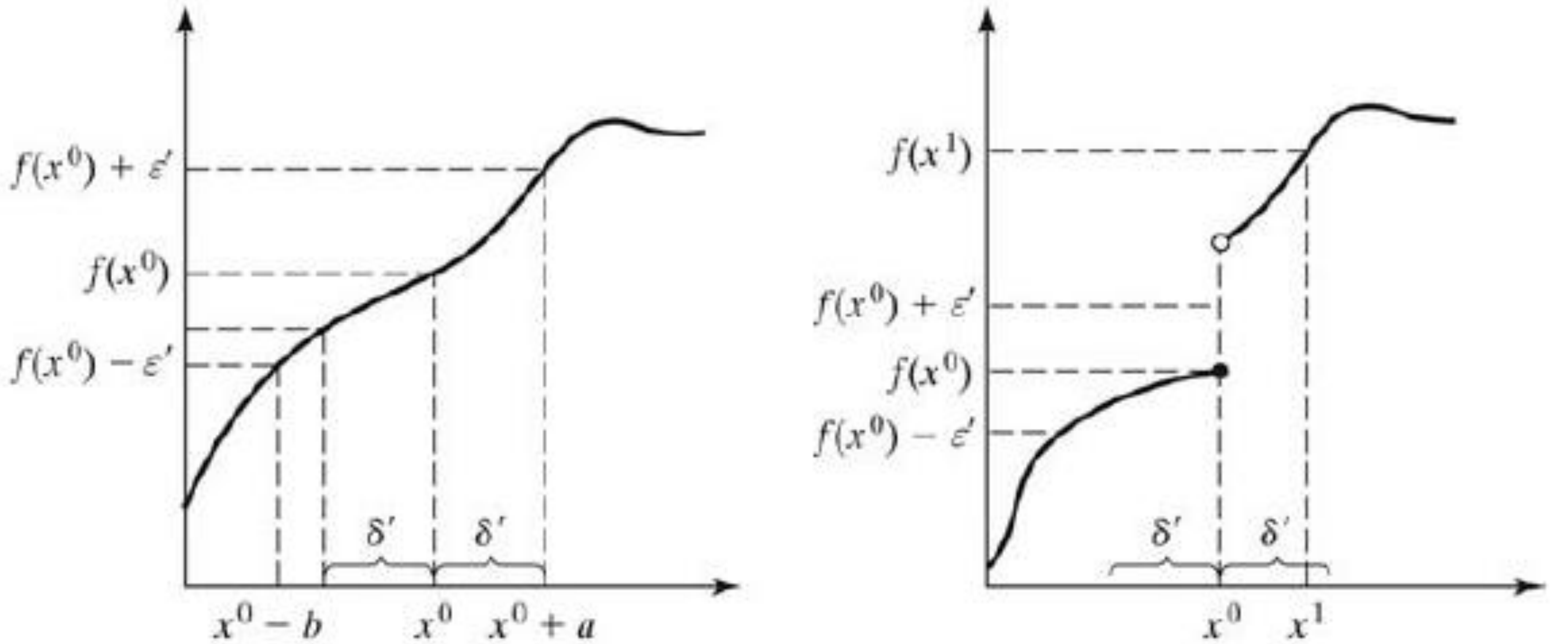
1. *The empty set, $\emptyset$, is a closed set.*
2. *The entire space, $\mathbb{R}^n$, is a closed set.*
3. *The union of any finite collection of closed sets is a closed set.*
4. *The intersection of closed sets is a closed set.*

Bounded Sets

A set S in $\mathbb{R}^n$ is called bounded if it is entirely contained within some ε-ball (either open or closed). That is, S is bounded if there exists some $\varepsilon > 0$ such that $S \subset B_\varepsilon(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^n$.

Continuity

A function $f: \mathbb{R} \to \mathbb{R}$ is *continuous at a point* x^0 if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x, x^0) < \delta$ implies that $d(f(x), f(x^0)) < \varepsilon$. A function is called a *continuous function* if it is continuous at every point in its domain.



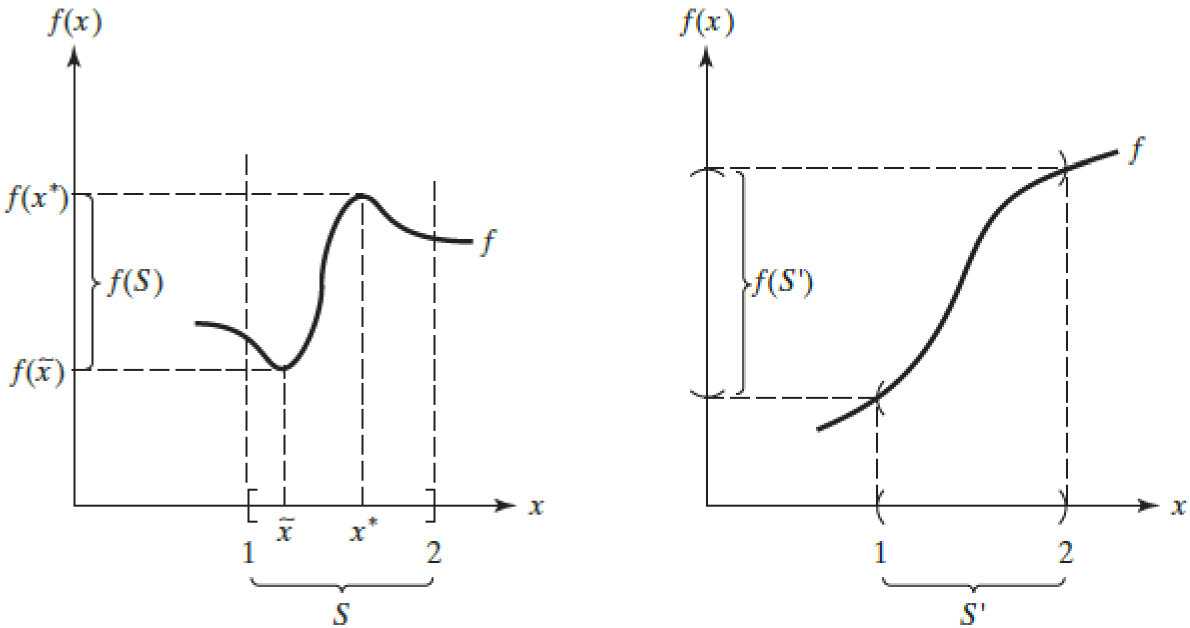
(a) (b)

Weierstrass Theorem

(Weierstrass) Existence of Extreme Values

Let $f: S \to \mathbb{R}$ be a continuous real-valued mapping where S is a non-empty compact subset of $\mathbb{R}^n$. Then there exists a vector $\mathbf{x}^* \in S$ and a vector $\tilde{\mathbf{x}} \in S$ such that

$$f(\tilde{\mathbf{x}}) \leq f(\mathbf{x}) \leq f(\mathbf{x}^*) \text{ for all } \mathbf{x} \in S.$$



(a) (b)

Real-Valued Functions

$f: D \to R$ is a real-valued function if D is any set and $R \subset \mathbb{R}$.

Increasing, Strictly Increasing and Strongly Increasing Functions

Let $f: D \to \mathbb{R}$, where D is a subset of $\mathbb{R}^n$. Then f is increasing *if $f(\mathbf{x}^0) \geq f(\mathbf{x}^1)$ whenever $\mathbf{x}^0 \geq \mathbf{x}^1$. If, in addition, the inequality is strict whenever $\mathbf{x}^0 \gg \mathbf{x}^1$, then we say that f is* strictly increasing. *If, instead, $f(\mathbf{x}^0) > f(\mathbf{x}^1)$ whenever $\mathbf{x}^0$ and $\mathbf{x}^1$ are distinct and $\mathbf{x}^0 \geq \mathbf{x}^1$, then we say that f is* strongly increasing.

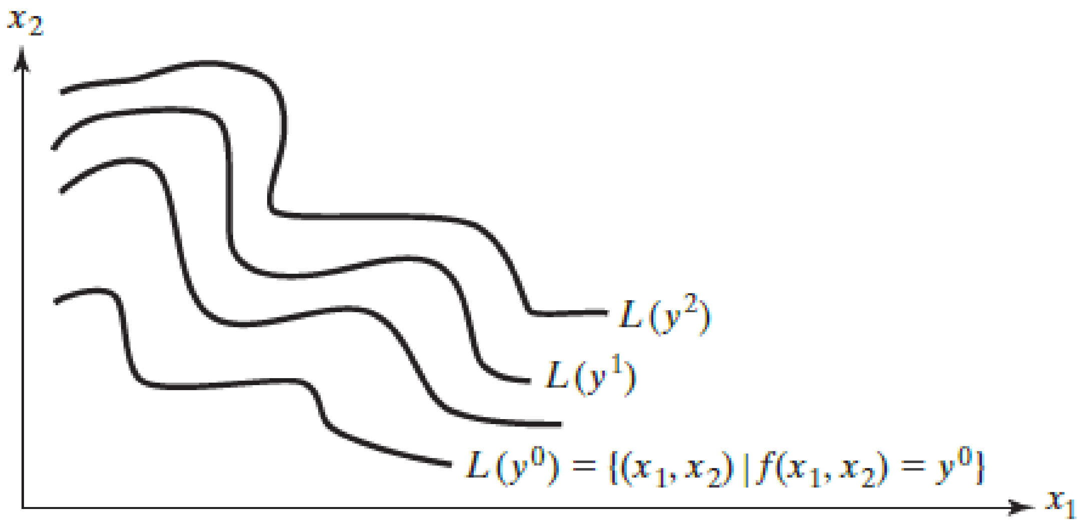
Decreasing, Strictly Decreasing and Strongly Decreasing Functions

Let $f: D \to \mathbb{R}$, where D is a subset of $\mathbb{R}^n$. Then f is decreasing *if $f(\mathbf{x}^0) \leq f(\mathbf{x}^1)$ whenever $\mathbf{x}^0 \geq \mathbf{x}^1$. If, in addition, the inequality is strict whenever $\mathbf{x}^0 \gg \mathbf{x}^1$, then we say that f is* strictly decreasing. *If, instead, $f(\mathbf{x}^0) < f(\mathbf{x}^1)$ whenever $\mathbf{x}^0$ and $\mathbf{x}^1$ are distinct and $\mathbf{x}^0 \geq \mathbf{x}^1$, then we say that f is* strongly decreasing.

Level Sets

Level Sets

$L(y^0)$ is a level set of the real-valued function $f: D \to R$ iff $L(y^0) = \{x \mid x \in D, f(x) = y^0\}$, where $y^0 \in R \subset \mathbb{R}$.



Superior and Inferior Sets

1. $S(y^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in D, f(\mathbf{x}) \geq y^0\}$ *is called the superior set for level* y^0.
2. $I(y^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in D, f(\mathbf{x}) \leq y^0\}$ *is called the inferior set for level* y^0.
3. $S'(y^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in D, f(\mathbf{x}) > y^0\}$ *is called the strictly superior set for level* y^0.
4. $I'(y^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in D, f(\mathbf{x}) < y^0\}$ *is called the strictly inferior set for level* y^0.

Concave Functions

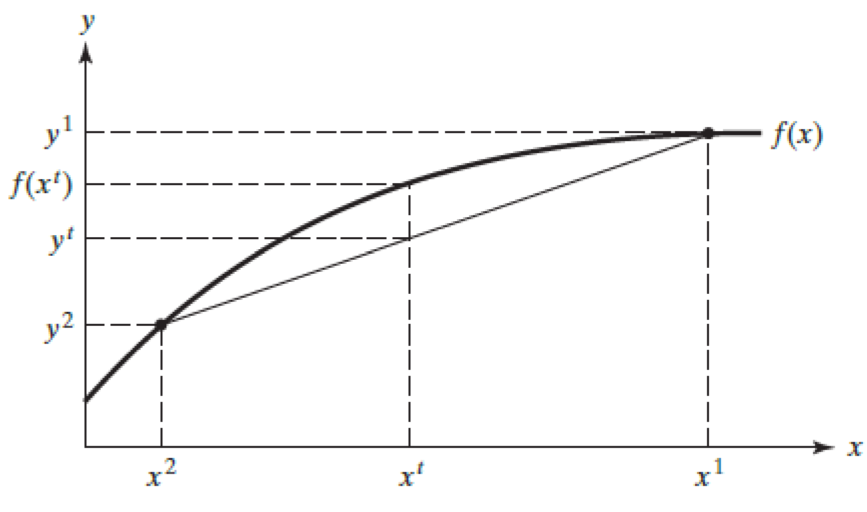
Real-Valued Functions Over Convex Sets

Throughout this section, whenever $f: D \to R$ is a real-valued function, we will assume $D \subset \mathbb{R}^n$ is a convex set. When we take $\mathbf{x}^1 \in D$ and $\mathbf{x}^2 \in D$, we will let $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^2$, for $t \in [0, 1]$, denote the convex combination of $\mathbf{x}^1$ and $\mathbf{x}^2$. Because D is a convex set, we know that $\mathbf{x}^t \in D$.

Concave Functions

$f: D \to R$ is a concave function if for all $\mathbf{x}^1, \mathbf{x}^2 \in D$,

$$f(\mathbf{x}^t) \geq tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \quad \forall\, t \in [0, 1].$$



Points On and Below the Graph of a Concave Function Form a Convex Set

Let $A \equiv \{(\mathbf{x}, y) \mid \mathbf{x} \in D, f(\mathbf{x}) \geq y\}$ *be the set of points 'on and below' the graph of* $f\colon D \rightarrow R$*, where* $D \subset \mathbb{R}^n$ *is a convex set and* $R \subset \mathbb{R}$*. Then,*

$$f \text{ is a concave function} \iff A \text{ is a convex set.}$$

Strictly Concave Functions

$f\colon D \rightarrow R$ *is a strictly concave function iff, for all* $\mathbf{x}^1 \neq \mathbf{x}^2$ *in* D*,*

$$f(\mathbf{x}^t) > tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \quad \text{for all } t \in (0, 1).$$

Quasiconcave Functions[7]

$f : D \to R$ is quasiconcave iff, for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in D,

$$f(\mathbf{x}^t) \geq \min[f(\mathbf{x}^1), f(\mathbf{x}^2)] \quad \text{for all } t \in [0, 1].$$

Quasiconcavity and the Superior Sets

$f : D \to \mathbb{R}$ is a quasiconcave function iff $S(y)$ is a convex set for all $y \in \mathbb{R}$.

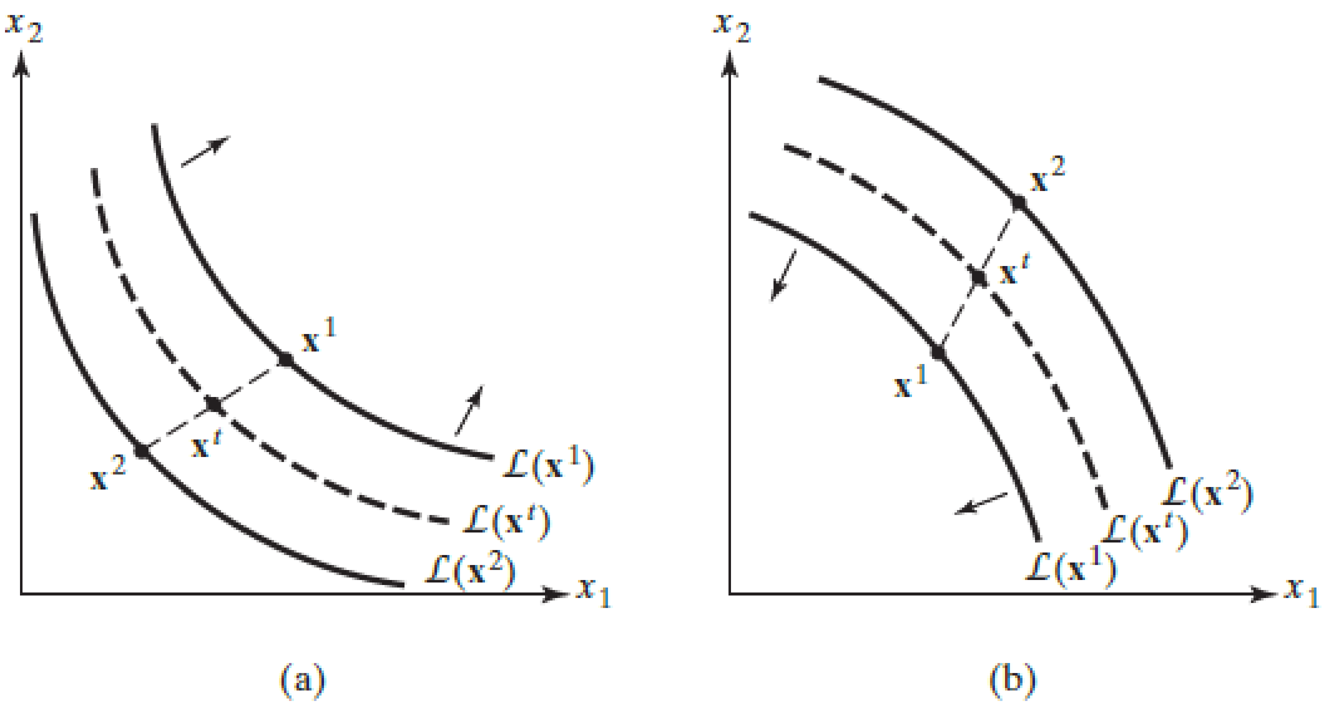


Figure A1. 30. Level sets for quasiconcave functions. (a) The function is quasiconcave and increasing. (b) The function is quasiconcave and decreasing.

Concavity Implies Quasiconcavity

A concave function is always quasiconcave. A strictly concave function is always strictly quasiconcave.

Convex and Strictly Convex Functions

1. $f: D \to R$ *is a convex function iff, for all* $\mathbf{x}^1, \mathbf{x}^2$ *in* D,

$$f(\mathbf{x}^t) \leq tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \text{ for all } t \in [0, 1].$$

2. $f: D \to R$ *is a strictly convex function iff, for all* $\mathbf{x}^1 \neq \mathbf{x}^2$ *in* D,

$$f(\mathbf{x}^t) < tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \text{ for all } t \in (0, 1).$$

Concave and Convex Functions

$f(\mathbf{x})$ *is a (strictly) concave function if and only if* $-f(\mathbf{x})$ *is a (strictly) convex function.*

Points On and Above the Graph of a Convex Function Form a Convex Set

Let $A^* \equiv \{(\mathbf{x}, y) \mid \mathbf{x} \in D, f(\mathbf{x}) \leq y\}$ *be the set of points 'on and above' the graph of* $f: D \to R$, *where* $D \subset \mathbb{R}^n$ *is a convex set and* $R \subset \mathbb{R}$. *Then*

$$f \text{ is a convex function} \iff A^* \text{ is a convex set.}$$

Quasiconvex and Strictly Quasiconvex Functions[8]

1. *A function* $f : D \to R$ *is quasiconvex iff, for all* $\mathbf{x}^1, \mathbf{x}^2$ *in D,*

$$f(\mathbf{x}^t) \leq \max[f(\mathbf{x}^1), f(\mathbf{x}^2)] \qquad \forall\, t \in [0, 1].$$

2. *A function* $f : D \to R$ *is strictly quasiconvex iff, for all* $\mathbf{x}^1 \neq \mathbf{x}^2$ *in D,*

$$f(\mathbf{x}^t) < \max[f(\mathbf{x}^1), f(\mathbf{x}^2)] \qquad \forall\, t \in (0, 1).$$

Quasiconvexity and the Inferior Sets

$f : D \to R$ *is a quasiconvex function iff* $I(y)$ *is a convex set for all* $y \in \mathbb{R}$.

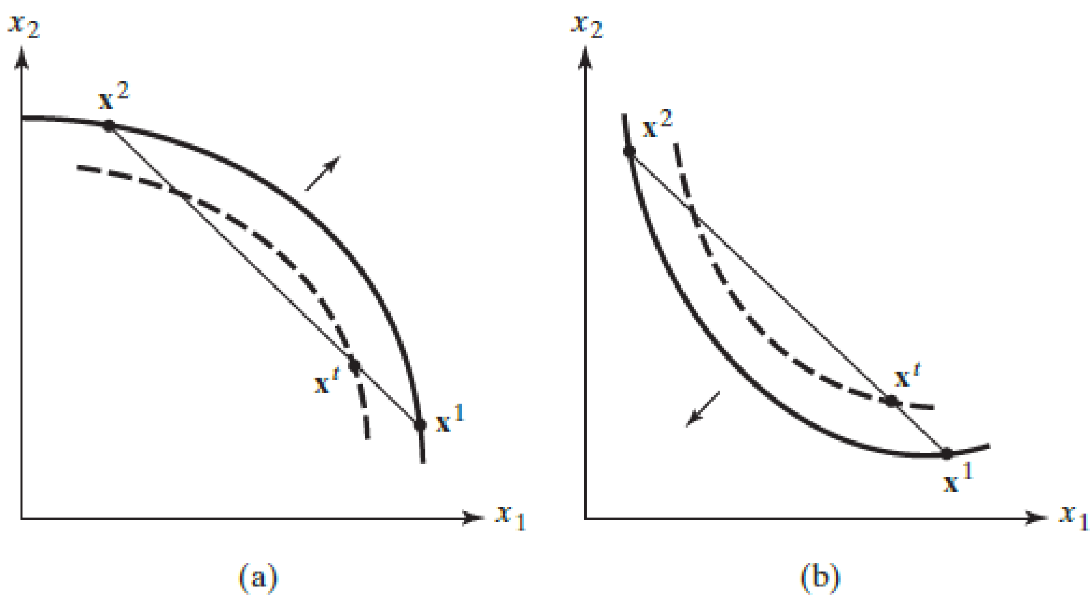


Figure A1. 34. Quasiconvex functions have convex inferior sets. Strictly quasiconvex functions have no linear segments in their level sets. (a) Strictly quasiconvex and increasing. (b) Strictly quasiconvex and decreasing.

Summary

f is concave	$\Longleftrightarrow$	the set of points *beneath* the graph is convex
f is convex	$\Longleftrightarrow$	the set of points *above* the graph is convex
f quasiconcave	$\Longleftrightarrow$	superior sets are convex sets
f quasiconvex	$\Longleftrightarrow$	inferior sets are convex sets
f concave	$\Rightarrow$	f quasiconcave
f convex	$\Rightarrow$	f quasiconvex
f (strictly) concave	$\Longleftrightarrow$	$-f$ (strictly) convex
f (strictly) quasiconcave	$\Longleftrightarrow$	$-f$ (strictly) quasiconvex

Calculus

Functions of a single variable

Concavity and First and Second Derivatives

Let D be a non-degenerate interval of real numbers on the interior of which f is twice continuously differentiable. The following statements 1 to 3 are equivalent:

1. *f is concave.*
2. *$f''(x) \leq 0 \quad \forall$ non-endpoints $x \in D$.*
3. *For all $x^0 \in D$: $f(x) \leq f(x^0) + f'(x^0)(x - x^0) \quad \forall\, x \in D$.*

Moreover,

4. *If $f''(x) < 0 \quad \forall$ non-endpoints $x \in D$, then f is strictly concave.*

Functions of several variables

$$H(\mathbf{x}) = \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \dots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \dots & f_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \dots & f_{nn}(\mathbf{x}) \end{pmatrix}$$

The Hessian of a function of several variables

Young's Theorem

For any twice continuously differentiable function $f(\mathbf{x})$,

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_i} \quad \forall\ i \text{ and } j.$$

➔ **The Hessian** is symmetric

Homogeneous Functions

Homogeneous Functions

A real-valued function $f(\mathbf{x})$ is called homogeneous of degree k if

$$f(t\mathbf{x}) \equiv t^k f(\mathbf{x}) \qquad \text{for all } t > 0.$$

Two special cases are worthy of note: $f(\mathbf{x})$ is homogeneous of degree 1, or linear homogeneous, if $f(t\mathbf{x}) \equiv tf(\mathbf{x})$ for all $t > 0$; it is homogeneous of degree zero if $f(t\mathbf{x}) \equiv f(\mathbf{x})$ for all $t > 0$.

Euler's Theorem

$f(\mathbf{x})$ is homogeneous of degree k if and only if

$$kf(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i \qquad \text{for all } \mathbf{x}.$$

Unconstrained Optimization

Necessary Conditions for Local Interior Optima in the Single-Variable Case

Let $f(x)$ be a twice continuously differentiable function of one variable. Then $f(x)$ reaches a local interior

$$
\begin{array}{llll}
1. & \text{maximum at } x^* & \Rightarrow f'(x^*) = 0 & (FONC), \\
 & & \Rightarrow f''(x^*) \leq 0 & (SONC). \\
2. & \text{minimum at } \tilde{x} & \Rightarrow f'(\tilde{x}) = 0 & (FONC), \\
 & & \Rightarrow f''(\tilde{x}) \geq 0 & (SONC).
\end{array}
$$

First-Order Necessary Condition for Local Interior Optima of Real-Valued Functions

If the differentiable function $f(\mathbf{x})$ reaches a local interior maximum or minimum at $\mathbf{x}^*$, then $\mathbf{x}^*$ solves the system of simultaneous equations,

$$
\begin{aligned}
\frac{\partial f(\mathbf{x}^*)}{\partial x_1} &= 0 \\
\frac{\partial f(\mathbf{x}^*)}{\partial x_2} &= 0 \\
&\vdots \\
\frac{\partial f(\mathbf{x}^*)}{\partial x_n} &= 0.
\end{aligned}
$$

Unconstrained Optimization

Second-Order Necessary Condition for Local Interior Optima of Real-Valued Functions

Let $f(\mathbf{x})$ be twice continuously differentiable.

1. *If $f(\mathbf{x})$ reaches a local interior maximum at $\mathbf{x}^*$, then $\mathrm{H}(\mathbf{x}^*)$ is negative semidefinite.*
2. *If $f(\mathbf{x})$ reaches a local interior minimum at $\tilde{\mathbf{x}}$, then $\mathrm{H}(\tilde{\mathbf{x}})$ is positive semidefinite.*

Strict Concavity/Convexity and the Uniqueness of Global Optima

1. *If $\mathbf{x}^*$ maximises the strictly concave function f, then $\mathbf{x}^*$ is the unique global maximiser, i.e., $f(\mathbf{x}^*) > f(\mathbf{x})\ \forall\ \mathbf{x} \in D,\ \mathbf{x} \neq \mathbf{x}^*$.*
2. *If $\tilde{\mathbf{x}}$ minimises the strictly convex function f, then $\tilde{\mathbf{x}}$ is the unique global minimiser, i.e., $f(\tilde{\mathbf{x}}) < f(\mathbf{x})\ \forall\ \mathbf{x} \in D,\ \mathbf{x} \neq \mathbf{x}^*$.*

Constrained Optimization w/ Equality Constrains

$$\max_{x_1, x_2} f(x_1, x_2) \qquad \text{subject to} \qquad g(x_1, x_2) = 0.$$

$$\mathcal{L}(x_1, x_2, \lambda) \equiv f(x_1, x_2) - \lambda g(x_1, x_2).$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} - \lambda^* \frac{\partial g(x_1^*, x_2^*)}{\partial x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} - \lambda^* \frac{\partial g(x_1^*, x_2^*)}{\partial x_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) = 0.$$

Envelope Theorem

Describes how the optimal value of the objective function in a parametrized optimization problem changes as one of the parameters changes

Let $f, h_1, ..., : R^n \times R^1 \to R^1$ be a C^1 functions. Let $\mathbf{x}^*(a) = (x_1^*(a), x_2^*(a), ..., x_n^*(a))$

denote the solution of the problem of:

$\max f(\mathbf{x}, a)$

$s.t.$

$h_i(\mathbf{x}, a) = 0;$ i = 1, ...,k

for any fixed choice of the paramter a.

Suppose that $\mathbf{x}^*(a)$ and the Lagrange multipliers $\mu_1(a), ..., \mu_n(a)$ are C^1 functions and that the NDCQ holds. Then:

$$\frac{d}{da} f(\mathbf{x}^*(a), a) = \frac{\partial L}{\partial a}(\mathbf{x}^*(a), \mu(a), a),$$

where L is teh natural Lagrangian for this problem.

Comparative Statics

Consider a system of equations with 2 unknowns (*x* and *y*) and a parameter *t*:

$$f\left(x, y, t\right) = 0$$

$$g\left(x, y, t\right) = 0$$

Determine

$$\frac{dx}{dt}$$

Inverse Function Theorem

Let f be a C^1 defined on an interval I in R^1. If $f'(x) \neq 0$ for all $x \in I$, then

$a.$) f is invertible on I

$b.$) its inverse g is a C^1 function on the interval $f(I)$

$c.$) for all z in the domain of the inverse function g

$$g'(z) = \frac{1}{f'(g(z))}$$