Uniqueness of Coalitional Equilibria
CORRECTIONS, COMMENTS AND FURTHER READING

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Corrections:

1. *Page 2, line 6* $\downarrow$*:* ... If each strategy set X^i is a compact and convex subset of a real Banach space, each payoff function ...
2. *Page 2, line 15* $\downarrow$*:* $\mathbf{R}_\mathcal{C}$ is proper, convex-valued, closed-valued ...
3. *Page 2, line 16* $\downarrow$*:* applying the fixed point theorem of Bohnenblust and Karlin.
4. *Page 2, line 21* $\downarrow$*:* ... with respect to x^i exists ...
5. *Page 3, condition 2 of Theorem 4:* If p is differentiable with $p' < 0$ and each cost function is differentiable and strictly convex, then the game has at most one interior $\mathcal{C}$-equilibrium.
6. *Page 3, lines 3–6* $\uparrow$*:* ... is concave. To see it is, we first note we first note that the function $q \leadsto p(z+q)q$ (on $\sum_{l \in S} X^l \subseteq \mathbb{R}_+$) is a decreasing concave function of q multiplied by q which is known to be also concave. So the first sum being a composition of the linear function $\mathbf{a}^{\mathcal{C}_i} \mapsto \sum_{l \in \mathcal{C}_i} a^l$ with that concave function also is concave.
7. *Page 3, line 1* $\uparrow$*:* $B_K := \{\mathbf{a}^S \in R^S \mid \sum_{l \in S} a^l = K\}$. ...
8. *Page 4, line 2* $\downarrow$*:* ... convex subset of $\mathbb{R}^S$ ...
9. *Page 4, line 9* $\downarrow$*:* $c^{i\prime}(m^i(K_1)) < c^{i\prime}(m^i(K_2))$ for all i.
10. *Page 4, line 17* $\uparrow$*:* ... $= p'(y_\star) w^S_\star +$ ...
11. *Page 4, line 13* $\uparrow$*:* Because the function $\mathcal{K}' \to \mathbb{R}$ defined by

Comments: Theorem 2(2) even holds without assuming that φ is strictly increasing. Here is the new version:

Theorem 2 Consider a game in strategic form Γ where each strategy set X^i is an interval of $\mathbb{R}$ containing more than one point. Fix a coalition structure $\mathcal{C}$. Suppose for each $S \in \mathcal{C}$ and $i \in S$ that the partial derivative of the function F^S with respect to x^i exists as an element of $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Furthermore, suppose there exists an increasing function $\varphi : \mathbf{X} \to \mathbb{R}$ and with $Y := \varphi(\mathbf{X})$, for each $S \in \mathcal{C}$ and $i \in S$ a function $\mathcal{T}^i_S : X^i \times Y \to \overline{\mathbb{R}}$ that is strictly decreasing in its first and decreasing in its second variable such that for each $\mathbf{x} \in \mathbf{X}$

$$\frac{\partial F^S}{\partial x^i}(\mathbf{x}) = \mathcal{T}^i_S(x^i, \varphi(\mathbf{x}))$$

holds. Then, there exists at most one $\mathcal{C}$-equilibrium. $\diamond$

Proof.– Let $\mathbf{x}_*$ and $\mathbf{x}_\bullet$ be $\mathcal{C}$-equilibria. We may suppose that $y_* := \varphi(\mathbf{x}_*) \geq \varphi(\mathbf{x}_\bullet) \ =: \ y_\bullet$.

First, we prove that for all $S \in \mathcal{C}$ and $i \in S$ the inequality $x^i_* \leq x^i_\bullet$ holds. If $x^i_* = \inf X^i$ or $x^i_\bullet = \sup X^i$, then this result holds. Otherwise, x^i_* is not a left boundary point of X^i and $x^i_\bullet$ is not a right boundary point of X^i. Because $\mathbf{x}_*$ is a $\mathcal{C}$-equilibrium, $\mathbf{x}^S_*$ is a maximizer of the function $F^S_{\mathbf{x}^{\hat{S}}_*}$. This implies that x^i_* is a maximizer of the function $x^i \mapsto F^S(x^i; \mathbf{x}^{\hat{i}}_*)$ and therefore it follows that $0 \leq \frac{\partial F^S}{\partial x^i}(\mathbf{x}_*) = \mathcal{T}^i_S(x^i_*, y_*)$. By the same token, $0 \geq \frac{\partial F^S}{\partial x^i}(\mathbf{x}_\bullet) = \mathcal{T}^i_S(x^i_\bullet, y_\bullet)$. Therefore, $\mathcal{T}^i_S(x^i_*, y_*) \geq \mathcal{T}^i_S(x^i_\bullet, y_\bullet)$. Because $y_\bullet \leq y_*$, we have $\mathcal{T}^i_S(x^i_*, y_\bullet) \geq \mathcal{T}^i_S(x^i_*, y_*)$. Thus, $\mathcal{T}^i_S(x^i_*, y_\bullet) \geq \mathcal{T}^i_S(x^i_\bullet, y_\bullet)$. Because $\mathcal{T}^i_S$ is strictly decreasing in x^i we have $x^i_* \leq x^i_\bullet$. Now we even may conclude that $\mathbf{x}_* \leq \mathbf{x}_\bullet$ and thus $\varphi(\mathbf{x}_*) \leq \varphi(\mathbf{x}_\bullet)$. Now, as above, for all $S \in \mathcal{C}$ and $i \in S$ the inequality $x^i_\bullet \leq x^i_*$ holds. Thus $\mathbf{x}_\bullet = \mathbf{x}_*$. $\square$

<u>Further reading:</u>

M. Finus, P. v. Mouche and B. Rundshagen. On Uniqueness of Coalitional Equilibria. In: Contributions to Game Theory and Management. Volume VII, 51-60, 2014. Editors: L. Petrosjan, N. Zenkevich. St. Petersburg State University. ISSN 2310-2608.

If you think that some other things should be added here, then please let me know.