Advanced Microeconomics(ECH 32306)

*Homework 3 --- **Solutions***

Expected Utility Theory

1. On p.100 Jehle and Reny say that AXIOM G4 (Monotonicity) implies $a_1 \succ a_n$. Prove this.

Answer

We prove this by contradiction. Suppose $a_1 \sim a_n$, then $a_1 \succsim a_n$ and $a_n \succsim a_1$.
Furthermore, suppose we take $\alpha > \beta$.
Then by MON $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ but also $(\alpha \circ a_n, (1-\alpha) \circ a_1) \succsim (\beta \circ a_n, (1-\beta) \circ a_1)$.
The latter can be rearranged $((1-\alpha) \circ a_1, \alpha \circ a_n) \succsim ((1-\beta) \circ a_1, \beta \circ a_n)$ which by MON implies $1-\alpha \geq 1-\beta$ which implies $\beta \geq \alpha$ which contradicts the assumption.

2. Which of the following utility functions have the DARA property, that is, the Arrow-Pratt measure of absolute risk aversion R_a is decreasing in wealth.

(i) $u(w) = \ln w$
(ii) $u(w) = -a(w^2 - 2bw)$; $a, b > 0$
(iii) $u(w) = 1 - e^{1-w}$
(iv) $u(w) = aw^{\alpha}$; $a > 0, \alpha > 0$
(v) Discuss why or why not DARA is a convincing property.

Answer

$u(w)$	u'	u''	$R_a = -u''/u'$	DARA
$u(w) = \ln(w)$; $w_0 > 0$	$\frac{1}{w}$	$-\frac{1}{w^2}$	$\frac{1}{w}$	Yes
$u(w) = -a(w^2 - 2bw)$; $a > 0$	$-a(2w - 2b)$	$-2a$	$-\frac{1}{w-b}$	No for $w \leq b$
$u(w) = 1 - e^{1-w}$	e^{1-w}	$-e^{1-w}$	1	No
$u(w) = aw^{\alpha}$; $a > 0, \alpha > 0$	$\alpha a w^{\alpha-1}$	$(\alpha-1)\alpha a w^{\alpha-2}$	$-(\alpha-1)\frac{1}{w}$	Yes for $\alpha < 1$, else no

(v) A decision maker for whom DARA is violated would be willing to pay higher risk premiums the richer he gets. That is not very convincing.

3. A sports fan has an expected utility function $u(w) = \ln w$. She has subjective probability p that her favourite club "Ajax" will win the next match and probability $1-p$ that they will not win. She chooses to bet €x on Ajax such that if Ajax wins, she wins €x, but if Ajax does not win she loses €x. The fan's initial wealth is w_0.

(a) Determine the fan's degree of absolute and relative risk aversion.
(b) How can you determine the fan's subjective odds $p/(1-p)$ by observing the size of his bet?
(c) How much would she bet if $p = \frac{3}{5}$ and $w_0 = 9$?

Now suppose for the next game the fan has received a betting ticket from a friend as gift. If Ajax wins, she receives €2x, but if Ajax does not win, her loss is already paid for. This time $p = 1/2$.

(d) For how much would the fan sell her betting ticket depending on x and w_0?
(e) Calculate the subjective money value of the gift when $w_0 = 9$ and $x = 1$.

Answer

a) The agent is risk averse $R_a = \frac{1}{w}; R_r = 1$.

b) A risk neutral player would bid any amount if $\frac{p}{1-p} > 1$. A risk averse agent balances "trust" and risk aversion. Hence the agent maximizes.

$V = pu(w_0 + x) + (1-p)u(w_0 - x)$. From the first order conditions we obtain $\frac{p}{1-p} = \frac{w_0+x}{w_0-x}$.

c) Using the latter formula, we can use the numbers to obtain $x = \frac{9}{5}$.

d) Now we need to calculate the certainty equivalent value w_c of the value of the lottery ticket t when eventual losses are paid for.

$w_c = \ln(w_0 + t) = p\ln(w_0 + 2x) + (1-p)\ln(w_0)$.

Using $p = 1/2$ we get

$$\ln(w_0 + t) = \tfrac{1}{2}(\ln(w_0 + 2x) + \ln(w_0))$$
$$\Leftrightarrow 2\ln(w_0 + t) = \ln(w_0^2 + 2w_0 x)$$
$$\Leftrightarrow (w_0 + t)^2 = w_0^2 + 2w_0 x$$

Solving for t gives

$t = -w_0 + \sqrt{w_0^2 + 2xw_0}$.

e) Using the numbers we can calculate $t = -9 + \sqrt{81+18} \approx 0.95$.

Value of Information

4. Consider a chemical. Its use gives a benefit

$$B(x) = x^{\beta},$$

where x is the level of exposure, β ($0 < \beta < 1$) is a substance specific parameter. In addition the use of the substance gives a damage $D(x, \tau) = x\tau$ that depends on exposure and the substance's toxic potential τ. Suppose, through adequate measures the regulator can control exposure.

(a) How should the regulator set exposure of a substance with known toxic potential?
(b) Now suppose the toxic potential is unknown but known to be either low τ_l or high τ_h with probabilities p and $1-p$, respectively. How should exposure be regulated?
(c) Consider case (b) and assume $\tau_l = 0.1$, $\tau_h = 10$, $\beta = 0.5$, and $p = 2/3$. Calculate optimal exposure and the expected welfare (benefits minus damage cost)
(d) Using the specification in (c), what is the regulators willingness to pay for a test that provides perfect information about the substance's toxic potential?

Answer

(a) Equate the marginal benefit and the marginal cost:
$x = (\frac{1}{\beta}\tau)^{\frac{1}{\beta-1}}$

(b) Now we equate marginal benefit with the expected marginal damage
$\beta x^{\beta-1} = (p\tau_l + (1-p)\tau_h)^{\frac{1}{\beta-1}}$ and solve for *x*.

(c) Using the specification we obtain $x_{\exp} = (\frac{15}{102})^2 = 0.0216$. The associated expected welfare is $B(x_{\exp}) - D(x_{\exp}, \tau) = 0.218$.

(d) If the substance were known to be $\tau_l = 0.1$, then the optimal emission level is $x_l = 25$ and the associated welfare is $B(x_l) - D(x_l, \tau_l) = 2.5$
If the substance were known to be $\tau_h = 10$, then the optimal emission level is $x_h = 0.0025$ and the associated welfare is $B(x_h) - D(x_h, \tau_h) = 0.025$.
So, if we can act on perfect information we get either 2.5 or 0.025
The former situation would occur with $p = 2/3$ and the latter with $1-p = 1/3$. The resulting expected value is 1.675. The expected value of acting without further information, i.e. acting on prior beliefs was 0.218 as calculated before.
The value of information is thus $VOI = 1.675 - 0.218 = 1.457$

5. Let $A = \{a_1, a_2, a_3\}$ where $a_1 \succ a_2 \succ a_3$. A gamble *g* offers a_2 with certainty. Prove that if $a_2 \sim (\alpha \circ a_1, (1-\alpha) \circ a_3)$, then α must be strictly between zero and one.

Proof: By Continuity we know that $\alpha \in [0,1]$. Suppose $\alpha = 0$. Then $a_2 \sim (0 \circ a_1, 1 \circ a_3) = a_3$ which contradicts the assumption.
Now suppose suppose $\alpha = 1$. Then $a_2 \sim (1 \circ a_1, 0 \circ a_3) = a_1$ which also contradicts the assumption. Hence $\alpha \in (0,1)$.

Games with incomplete Information

6. *The ultimatum game*. Consider the following two-player game. Player 1 receives *x* coins of 1€ and *x* is a strictly positive even number. Player 1 offers none or a positive number of these coins to player 2. If player 2 rejects the offer, all the money is withdrawn. If player 2 accepts the offer, she receives what is offered to her and player 1 keeps the rest.

(a) Describe in a formal way the strategy spaces of the players and draw a game tree.

(b) What type of game is this and what is an appropriate solution concept?

In experiments many people reject strictly positive offers in the ultimatum game if these are considered "unfair". Suppose $\frac{1}{3}$ of the population has strong fairness preferences such that if they were in the role of player 2, they would reject any offer which is worth less than $\frac{1}{2}x$.

(c) What type of game is this and what is an appropriate solution concept? Draw the game tree (Hint: Harsanyi introduced a method to represent this kind a game).

(d) Derive the equilibrium.

(e) Determine the threshold fraction of "fair" players in the population such that only "fair" offers occur in equilibrium.

The ultimatum game is a famous game from the early days of experimental economics (W Güth, R Schmittberger, B Schwarze (1982) An Experimental Analysis of Ultimatum Bargaining. Journal of Economic Behavior and Organisation 3, 367-388.)

Answer

a) strategy space of player 1 is $\{0,1,...,x\}$ and for player 2 $\{\text{accept, reject}\}$.
b) Sequential perfect information game,
Solution concept: Subgame Perfect Nash Equilibrium
c) Sequential incomplete information game. Sequential Equilibrium. Harsanyi introduced the idea to represent the uncertainty about the type of the players with a move by nature. So with probability 1/3 player 2 is of type "fair" and with probability 2/3 the player is of type "rational". In the game tree Nature moves first choosing "fair" or "rational". This leads to two nodes in the same information set of player 1 who determines the offer from $\{0,1,...,x\}$. Player 2, fair or rational and knowing her own type sees the offer and accepts or rejects.
d) first note that 1 never offers more than *x*/2. If the type of 2 were known, 1 would offer *x*/2 to a fair player and 1 to a rational player (who strictly prefers 1 over nothing and would accept).
Hence 1 offers either 1 or *x*/2. The former yields $\frac{1}{3}\cdot 0+\frac{2}{3}\cdot(x-1)$, the latter yields *x*/2 regardless of type. Hence, play "*x*/2" if $\frac{x}{2}\geq\frac{2}{3}\cdot(x-1)\Leftrightarrow x\leq 4$. Else play "1".
e) Generally if *p* is 1's belief that player 2 is "fair", then offering 1 yields $p\cdot 0+(1-p)\cdot(x-1)$ and offering "*x*/2" yields $\frac{x}{2}$. The condition for offering "*x*/2" is $\frac{x}{2}\geq(1-p)\cdot(x-1)\Leftrightarrow\frac{x-2}{2(x-1)}\leq p$.

7. *A principle-agent game*: There are two agents: a risk neutral land owner and a risk averse farmer. Harvest is subject to risk. The risk is impacted be the farmer's effort. The land owner typically cannot observe the farmer's effort. Only harvest is observable.

a) Argue why or why not the land owner should pay a fixed wage to the farmer.

Assume now that effort *e* is either low $e_l=0$ or high $e_h=1$. With low effort harvest is either low $x_l=15$ with probability $\frac{2}{3}$ or high $x_h=60$ with probability $\frac{1}{3}$. With high effort harvest is low with probability $\frac{1}{3}$ or high with probability $\frac{2}{3}$. Furthermore assume the farmer can earn an off-farm wage $w_0 = 30.25$ with effort $e_0 = 0.5$. The landowner maximises profits (assume the price of the crop is 1). The farmer's utility function is $u(w,e) = \sqrt{w} - e$.

b) Give a formal description of the land owner's maximisation problem. What are the relevant constraints the land owner faces when offering a contract to the farmer? Describe the contract that the land owner offers to the farmer.

c) Will this contract be accepted? Calculate the land owner's profit if the contract is accepted.

Answer

a) Argue why or why not the land owner should pay a fixed wage to the farmer.

When the landowner pays a fixed wage he bears all the risk which is the efficient risk allocation. However, in that case, there is no incentive for the farmer to work hard. Effort will be low and the landowner receives lower returns from the land than with an incentive contract.

Assume now that effort e is either low $e_l = 0$ or high $e_h = 1$. With low effort harvest is either low $x_l = 15$ with probability $\frac{2}{3}$ or high $x_h = 60$ with probability $\frac{1}{3}$. With high effort harvest is low with probability $\frac{1}{3}$ or high with probability $\frac{2}{3}$. Furthermore assume the farmer can earn an off-farm wage $w_0 = 30.25$ with effort $e_0 = 0.5$. The landowner maximises profits (assume the price of the crop is 1). The farmer's utility function is $u(w,e) = \sqrt{w} - e$.

b) Give a formal description of the land owner's maximisation problem. What are the relevant constraints the land owner faces when offering a contract to the farmer? Describe the contract that the land owner offers to the farmer.

Note that the reservation utility is 5. With low effort of the agent the landowner will have to pay at least $w_0 = 25$. The returns are $\frac{2}{3}\cdot 15 + \frac{1}{3}\cdot 60 = 30$, so the land owner will make a profit of 5 when the agent's effort is low.
Next we examine the land owner's profits from high effort. He maximises return from high effort minus wage subject to a participation and an incentive constraint.

$$\max_{w_l, w_h} \tfrac{1}{3}\cdot 15 + \tfrac{2}{3}\cdot 60 - \tfrac{1}{3}\cdot w_l - \tfrac{2}{3}\cdot w_h$$

$$s.t.$$

$$PC: \tfrac{1}{3}\cdot\sqrt{w_l} + \tfrac{2}{3}\cdot\sqrt{w_h} - 1 \geq 5$$

$$IC: \tfrac{1}{3}\cdot\sqrt{w_l} + \tfrac{2}{3}\cdot\sqrt{w_h} - 1 \geq \tfrac{2}{3}\cdot\sqrt{w_l} + \tfrac{1}{3}\cdot\sqrt{w_h}$$

Rewriting the constraints we obtain the following Lagrangian function

$$L = \tfrac{1}{3}\cdot 15 + \tfrac{2}{3}\cdot 60 - \tfrac{1}{3}\cdot w_l - \tfrac{2}{3}\cdot w_h - \lambda(18 - \sqrt{w_l} - 2\sqrt{w_h}) - \mu(3 - \sqrt{w_h} + \sqrt{w_l})$$

From the first order necessary conditions we obtain:

(1) $\frac{\partial L}{\partial w_l} = 0 \Rightarrow \frac{2}{3}\sqrt{w_l} = \lambda - \mu$

(2) $\frac{\partial L}{\partial w_h} = 0 \Rightarrow \frac{2}{3}\sqrt{w_h} = \lambda + \frac{1}{2}\mu$.
Since $\lambda \geq 0, \mu \geq 0$ by definition, and we require

$\lambda(18 - \sqrt{w_l} - 2\sqrt{w_h}) = 0$

$\mu(3 - \sqrt{w_h} + \sqrt{w_l}) = 0,$

at least IC must be binding. If we rule out negative wages, PC must also be binding. With binding constraints we calculate $w_l = 16; w_h = 49;$ and expected profits are 7.

c) Will this contract be accepted? Calculate the land owner's profit if the contract is accepted.
The contract will be accepted because the IC holds. It is better than a fixed wage contract for low effort which yields profit $\frac{2}{3} \cdot 15 + \frac{1}{3} \cdot 60 - w_0 = 30 - 25 = 5$.

8. *Market for lemons*. Consider the following market for used cars. There are many sellers of used cars. Each seller has exactly one used car to sell and is characterised by the quality of the used car he wishes to sell. Let $\theta \in [0,1]$ index the quality of a used car and assume that θ is uniformly distributed on [0, 1]. If a seller of type θ sells his car (of quality θ) for a price *p*, his utility is $u_s(p, \theta)$. If he does not sell his car, then his utility is 0. Buyers of used cars receive utility $\theta - p$ if they buy a car of quality θ at price *p* and receive utility 0 if they do not purchase a car. There is asymmetric information regarding the quality of used cars. Sellers know the quality of the car they are selling, but buyers do not know its quality. Assume that there are not enough cars to supply all potential *k* buyers.

a) Argue that in a competitive equilibrium under asymmetric information, we must have $E(\theta \mid p) = p$.
b) Show that if $u_s(p, \theta) = p - \frac{\theta}{2}$, then every $p \in [0, \frac{1}{2}]$ is an equilibrium price.
c) Find the equilibrium price when $u_s(p, \theta) = p - \sqrt{\theta}$. Describe the equilibrium in words. In particular, which cars are traded in equilibrium?
d) Find an equilibrium price when $u_s(p, \theta) = p - \theta^3$. How many equilibria are there in this case?

Answer

(a) Suppose not. Could demand for used cars equal supply? There are two cases to consider. If $E(\theta \mid p) < p$, then the conditional average quality of cars on the market is below the price and buyers receive negative expected utility. So there is no demand and the market cannot clear. And if $E(\theta \mid p) > p$ then buyers receive positive expected utility and all potential buyers would like to buy but not all sellers would like to sell (those with high quality car will not sell as the price is less than average quality). So there is excess demand.
The equilibrium outcome is driven by positive selection: a higher price increases the average quality of the cars available on the market. So there are potentially many equilibria solving the pricing condition $E(\theta \mid p) = p$. There might exist a high price equilibrium where sellers put their high quality cars on the market and buyers are willing to pay the high price. There might also exist a low price equilibrium where the sellers remove the best cars from the market and buyers

are only willing to pay a low price.

(b) Fixing the price p, if $u_s(\theta \mid p) = p - \frac{\theta}{2}$, all sellers with quality $\theta \leq 2p$ will prefer selling their car to not selling. Thus the average quality of cars conditional on price p is given by

$E(\theta \mid p) = \min(\frac{2p}{2}, \frac{1}{2}) = p$

which is also the competitive equilibrium outcome. Thus any price $p \in [0, \frac{1}{2}]$ is an equilibrium with only cars of quality $\theta \leq 2p$ traded, so that the average quality is just equal to the price.

(c) When $u_s(\theta \mid p) = p - \sqrt{\theta}$, only sellers with quality $\theta \leq p^2$ will prefer to sell. Thus the quality threshold $\theta^* = p^2$ which yields the conditional average quality of cars $E(\theta \mid p) = \min(\frac{p^2}{2}, \frac{1}{2}) = p$ and $p = 0$ is the only possible solution.

(d) Similar to cases (b) and (c) we have $E(\theta \mid p) = \min(\frac{p^{1/3}}{2}, \frac{1}{2}) = p$. Such that we obtain $p = \sqrt{\frac{1}{8}}$.

Another equilibrium is $p = 0$.

Clearly there is market failure because high quality cars do not sell. Pareto improvements are not possible unless we change the game. One possibility is that sellers give guarantees.