# *Advanced Microeconomics*

*Part 3*

*a. Expected Utility Theory*

*b. Value of Information*

*c. Games with incomplete Information*

- *insurance*
- *the principal-agent problem*

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## *Make a choice*

$g_1 = \text{for sure } 1000$

or

$$g_2 = \begin{cases} \text{with } 10\% \text{ chance } 5000 \\ \text{with } 89\% \text{ chance } 1000 \\ \text{with } \;\; 1\% \text{ chance } 0 \end{cases}$$

Write down your choice.

## *Make a choice*

$$g_3 = \begin{cases} \text{with } 10\% \text{ chance } 5000 \\ \text{with } 90\% \text{ chance } 0 \end{cases}$$

or

$$g_4 = \begin{cases} \text{with } 11\% \text{ chance } 1000 \\ \text{with } 89\% \text{ chance } 0 \end{cases}$$

Write down your choice.

If you wish to compare the first choice was

$$g_1 = \text{for sure } 1000$$

$$g_2 = \begin{cases} \text{with } 10\% \text{ chance } 5000 \\ \text{with } 89\% \text{ chance } 1000 \\ \text{with } 1\% \text{ chance } 0 \end{cases}$$

# *Literature*

*Textbook*

- Jehle and Reny, section 2.4.

*Seminal works*


- Bernoulli, Daniel (1738 / 1954) Exposition of a new theory on the measurement of risk. (Translation by Louise Sommer from Latin: Specimen Theoriae Novae de Mensura Sortis. In: Commentarii Academiae Scientiarum Imperialis Petropolitanae V, 1738, 175-192.) Econometrica 22(1), 23-36.
- Neumann, John von / Morgenstern, Oskar (1944) Theory of Games and Economic Behavior. Second edition, Princeton 1947: Princeton University Press.
- Savage, Leonard (1954) The Foundations of Statistics. New York: Wiley.


*Additional reading*


- Machina, Mark J. (1989) Choice under Uncertainty: Problems Solved and Unsolved. Journal of Economic Perspectives 1(1), 121-154.
- Shaw, W. Douglass / Woodward, Richard T. (2008) Why environmental and resource economists should care about non-expected utility models. Resource and Energy Economics 30(1), 66-89.

# *Basic Concepts*

Expected Utility Theory deals with decision-making under risk.

*Basic Concepts*

- Outcomes: A finite set of outcomes (possible states of the world): $A = \{a_1, ..., a_n\}$.
- Probabilities: Each outcome $a_i$ occurs with probability $p_i$.
- Simple Gambles (lotteries): $g = (p_1 \circ a_1, ..., p_n \circ a_n)$

The decision-maker chooses from a set of lotteries. The choice is based on preferences.

- The set of simple gambles is $\mathcal{G}_S = \left\{ (p_1 \circ a_1, ..., p_n \circ a_n) \middle| p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}$.

If a lottery has another lottery as its prize, we have to deal with *compound gambles*.

Preferences are an ordering over the set of gambles.

## *An illustration*

A bank decides whether to give a mortgage to a customer or not.

- Outcomes: $A = \{\text{no loan}, \text{pays back with interest, bankrupt}\}$.
- Probabilities: $p_1, p_2, p_3$.

Two simple gambles:
The bank chooses between *giving the loan*

$$g_1 = (0 \circ \text{no loan}, \; p_2 \circ \text{pays back with interest}, \; (1 - p_2) \circ \text{bankrupt})$$

and *not giving the loan*

$$g_2 = (1 \circ \text{no loan}, 0 \circ \text{pays back with interest}, \; 0 \circ \text{bankrupt}).$$

A possible decision criterion is: maximise expected payoffs.

But … (see next slide)

# *The St. Petersburg Paradox (Daniel Bernoulli)*

The casino in St. Petersburg offers the following gamble:
You toss a coin until *heads* comes up. If *heads* comes up at the $n^{th}$ toss you receive a payoff of $2^n$ Rubels.
How much would you offer to participate in the gamble.

| | | | | |
|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | …. |
| payoff | 2 | 4 | 8 | … |
| prob. | 1/2 | 1/4 | 1/8 | … |
| exp. payoff | 1 | 1 | 1 | … |

The expected value of the lottery is infinity!

Bernoulli was the first to draw an important distinction …

# *Expected Utility Theory*

There is an important distinction to make between a money payoff and its worth. The St. Petersburg Paradox is resolved if value (utility) is a concave function of money payoff.

This idea has been employed by John von Neumann and Oskar Morgenstern in their *Theory of Games and Economic Behavior* (1944).
In the appendix to the second edition (1947) they develop axiomatic foundations for a theory for decision-making under risk.

The main assumptions:

- The decision-maker (DM) knows the possible states of the world.
- The DM knows the probability distribution attached to each choice option. (The latter assumption is relaxed in *Subjective Expected Utility Theory*.)

Common terminology is to refer to *risk* if the probability distribution is known and to refer to *uncertainty* (or *ambiguity*) if probabilities are not known. This distinction is due to F. Knight (1920).

# *Axioms for Expected Utility Theory (1)*

- Let $\mathcal{G}$ be a set of (simple and compound) gambles $g = (p_1 \circ g^1, ..., p_k \circ g^k)$.

The a preference ordering on $\mathcal{G}$ should satisfy the following axioms

Completeness (COM):

For any two gambles $g, g' \in \mathcal{G}$ either $g \succsim g'$ or $g' \succsim g$.

Transitivity (TRAN):

For any three gambles $g, g', g'' \in \mathcal{G}$, if $g \succsim g'$ and $g' \succsim g''$, then $g \succsim g''$.

Continuity (CON):

Suppose, without loss of generality, we label outcomes such that $a_1 \succsim a_2 \succsim ... \succsim a_n$.

For any gamble $g \in \mathcal{G}$ , there is some probability $\alpha$ such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$

Monotonicity (MON):

For all probabilities $\alpha, \beta \in [0,1]$,

$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ if and only if $\alpha \geq \beta$.

## *Axioms for Expected Utility Theory (2)*

Substitution (SUB):

If $g = (p_1 \circ g^1, ..., p_k \circ g^k)$ and $h = (p_1 \circ h^1, ..., p_k \circ h^k)$ and if $g^i \sim h^i$ for every $i$, then $g \sim h$.

Every compound gamble induces a unique simple gamble $g \in \mathcal{G}_S$.

Reduction (RED):

For any gamble $h \in \mathcal{G}$, if $g$ is induced by $h$, then $g \sim h$.

Theorem (von Neumann and Morgenstern):

Let preferences $\succcurlyeq$ over gambles satisfy axioms COM, TRAN, CON, MON, SUB and RED, then there exists a utility function $u : \mathcal{G} \rightarrow \mathbb{R}$ that represents the ordering $\succcurlyeq$ and has the expected utility property, i.e.

$$u(g) = \sum_{i=1}^{n} p_i u(a_i).$$

Oskar Morgenstern and John von Neumann (1946)

## *Implications for the utility concept*

Von Neuman-Morgenstern Utilities are cardinal utilities.

Ratios of utility differences are meaningful, i.e. unique for given preferences and given any three outcomes $a \succ b \succ c$. Let $\alpha$ be the (unique) probability such that $b \sim (\alpha \circ a, (1-\alpha) \circ c)$, then

$$\frac{u(a)-u(b)}{u(b)-u(c)} = \frac{1-\alpha}{\alpha}.$$

If $u$ represents $\succcurlyeq$, then for arbitrary numbers $\lambda$ and $\mu > 0$ the utility function $v = \mu u + \lambda$ represents the *same* preferences.

# *Risk attitudes*

Assume outcomes of a gamble *g* are levels of wealth *w*. Expected wealth is given by

$$E(g) \equiv \sum_{i=1}^{n} p_i w_i .$$

The DM is

- risk averse if $u(E(g)) > u(g)$,
- risk neutral if $u(E(g)) = u(g)$,
- risk loving if $u(E(g)) < u(g)$.

Consider a risk averse DM. There is a level of wealth $w^c$, labelled certainty equivalent, such that $u(w^c) = u(g)$. Hence, $u(E(g)) > u(w^c)$. We define the risk premium as the difference $P \equiv u(E(g)) - u(w^c)$.

## *Arrow-Pratt measure of Absolute Risk Aversion*

We introduce a notion of "more risk averse than" to compare preferences of different DMs (Pratt, Econometrica 1964; Arrow, 1970).

$R_a$ is the coefficient of absolute risk aversion.

$$R_a(w) \equiv -\frac{u''(w)}{u'(w)}.$$

$R_r$ is the coefficient of relative risk aversion.

$$R_r(w) \equiv -w\frac{u''(w)}{u'(w)}.$$

## *References*


- Arrow, Kenneth J. (1970) Essays in the Theory of Risk Bearing. Chicago: Markham.
- Pratt, John W. (1964) Risk Aversion in the Small and in the Large. Econometrica 32(1-2), 122-136.

# *What was your choice?*

$g_1 = \text{for sure } 1000$

$$g_2 = \begin{cases} \text{with 10\% chance } 5000 \\ \text{with 89\% chance } 1000 \\ \text{with } \ 1\text{\% chance } 0 \end{cases}$$

If you go for $g_1$, then EU theory requires that you choose $g_4$.
If you go for $g_2$, then EU theory requires that you choose $g_3$.

But many people don't!
This is Allais's paradox.

$$g_3 = \begin{cases} \text{with 10\% chance } 5000 \\ \text{with 90\% chance } 0 \end{cases}$$

or

$$g_4 = \begin{cases} \text{with 11\% chance } 1000 \\ \text{with 89\% chance } 0 \end{cases}$$

Allais, Maurice (1953) Le comportement de l'homme rationnel devant le risque: Critique des postulats et des axiomes de l'Ecole Américaine. Econometrica 21, 503-546.

*A useful survey*

- Schoemaker, P.J.H. (1982) The expected utility model: its variants, purposes, evidence and limitations. Journal of Economic Literature 20, 529-563.

*On the history of utility theory*

- Cooter, Robert D./Rappoport, Peter (1984) Were the Ordinalists Wrong About Welfare Economics? Journal of Economic Literature 22, 507-530.

*The behavioural challenge*

- Machina, Mark J. (1989) Choice under Uncertainty: Problems Solved and Unsolved. Journal of Economic Perspectives 1(1), 121-154.

- Loewenstein, George (1999) Because It Is There: The Challenge of Mountaineering … for Utility Theory. Kyklos 52, 315-343.

## *Subjective Expected Utility Theory*
## *The Ellsberg paradox*

Consider an urn with **30 red** balls and **60** balls that are **either black or yellow**.

Choose between:

$g_1 = €100$ if you draw "red"

$g_2 = €100$ if you draw "black"

And another draw from the same urn: choose between

$g_3 = €100$ if you draw "red" or "yellow"

$g_4 = €100$ if you draw "black" or "yellow"

## *Subjective Expected Utility Theory*
## *The Ellsberg paradox*

If $g_1 \succ g_2$, then you beliefs must be such that $p_{red} \succ p_{black}$.

But having that belief you should $g_3 \succ g_4$ because $p_{yellow}$ is the same for the two gambles.

| |
|---|
| But many people don't!<br>This is Ellsberg's paradox. |

Ellsberg, Daniel (1961) Risk, Ambiguity and the Savage Axioms. Quarterly Journal of Economics 75, 643-669.

## *Decision-making under uncertainty*

## *Part b: Value of information*

A simple benchmark model (e.g. Olson 1990)*:

*Scheme of payoffs from taking safety measures on the use of a chemical substance*

| | $\tau = 1$ | $\tau = 0$ |
|---|---|---|
| $\alpha = s$ | $a$ | $b$ |
| $\alpha = l$ | $c$ | $d$ |

With $d > b \geq a > c$

and a prior belief $p_0$ that the substance is toxic.

* Olson, Lars J. (1990) The Search for a Safe Environment: The Economics of Screening and Regulating Environmental Hazards. Journal of Environmental Economics and Management 19, 1-18.

Generally, the DM maximises $V \equiv \max_{\alpha} \mathrm{E}\left[U(\alpha,\tau)\right]$

Hence – with two actions and two states – the strict policy is preferred if and only if

strict policy $\succcurlyeq$ lenient policy , i.e.

$$p_0 a + (1 - p_0) b \geq p_0 c + (1 - p_0) d \quad \text{for } p_0 \in [0,1]$$

Value with prior information

$$V_0 = \max\left( p_0 a + (1 - p_0) b,\ p_0 c + (1 - p_0) d \right)$$

*Conditional probabilities* $prob(t_1 | \tau)$ *of seeing a positive* $t_1^+$ *or a negative* $t_1^-$ *test outcome for given toxicity*

| | $\tau = 1$ | $\tau = 0$ |
|---|---|---|
| $t_1^+$ | $\varphi$ probability of a true positive result "sensitivity" | $1-\psi$ probability of a false positive result |
| $t_1^-$ | $1-\varphi$ probability of a false negative result | $\psi$ probability of a true negative result "specificity" |

*Posterior beliefs of a substance being toxic (non-toxic) after evidence from test 1*

*Bayes's Rule*

| $p_1(\tau \mid t_1)$ $p_1(\tau \mid t_i)$ | $\tau = 1$ | $\tau = 0$ |
|---|---|---|
| $t_1^+$ | $p_1^+ = \dfrac{p_0\varphi}{p_0\varphi + (1-p_0)(1-\psi)}$ | $(1-p_1^+) = \dfrac{(1-p_0)(1-\psi)}{p_0\varphi + (1-p_0)(1-\psi)}$ |
| $t_1^-$ | $p_1^- = \dfrac{p_0(1-\varphi)}{p_0(1-\varphi) + (1-p_0)\psi}$ | $(1-p_1^-) = \dfrac{(1-p_0)\psi}{p_0(1-\varphi) + (1-p_0)\psi}$ |

Now, given the test result we need to distinguish two cases.

(a) For $t_1 = t_1^+$ (positive test result) we have

$$V_1^+ \equiv \max\left[ p_1^+ a + (1 - p_1^+)b,\ p_1^+ c + (1 - p_1^+)d \right].$$

(b) *For* $t_1 = t_1^-$ (negative test result) we have

$$V_1^- \equiv \max\left[ p_1^- a + (1 - p_1^-)b, p_1^- c + (1 - p_1^-)d \right]$$

The expected payoff of performing a test is the weighted average of $V_1^+$ and $V_1^-$, where the weights are given by the probability of a test being positive or negative. Thus we have

$$V_1 = \Pr_1^+ V_1^+ + \Pr_1^- V_1^-$$

with

$$\Pr_1^+ = p_0 \varphi + (1 - p_0)(1 - \psi) \text{ and } \Pr_1^- = p_0 (1 - \varphi) + (1 - p_0)\psi .$$

The expected value of information (*VOI*) is defined as the difference between the expected value from an optimal decision on the use of *s* with and without additional evidence from test *i*.

$VOI \equiv V_1 - V_0$.

Since testing is usually costly, test if and only if

$$VOI - k_1 \geq 0. \quad \text{(Under risk neutrality.)}$$

# *Example: Market research for a new product*

Action: produce or not produce
States of the World: small market or large market $(m = S, L)$

*Table of payoffs*

| | S | L |
|---|---|---|
| $\neg$ produce | $a = 0$ | $b = 0$ |
| produce | $c = -2$ | $d = 4$ |

Assume a prior belief $p_0 = \frac{1}{2}$ that the market is small.

Acting on prior believes: "produce". Then, $V_0 = 1$.

## *Market research*

Probabilities *of seeing a "small" or "large" market*

| | S | L |
|---|---|---|
| $R^S$ | $\varphi = 0.8$ | $1 - \psi = 0$ |
| $R^L$ | $1 - \varphi = 0.2$ | $\psi = 1$ |

*Posterior beliefs of a substance being toxic (non-toxic) after evidence from test 1*

*Bayes's Rule*

| $p_1(m \mid R^m)$ | S | L |
|---|---|---|
| $R^S$ | $p_1^S = \frac{p_0\varphi}{p_0\varphi + (1-p_0)(1-\psi)}$ | $(1-p_1^S)$ |
| $R^L$ | $p_1^L = \frac{p_0(1-\varphi)}{p_0(1-\varphi)+(1-p_0)\psi}$ | $(1-p_1^L)$ |

| $p_1(m \mid R^m)$ | S | L |
|---|---|---|
| $R^S$ | $p_1^S = \frac{0.4}{0.4+0}$ | 0 |
| $R^L$ | $p_1^L = \frac{0.1}{0.1+0.5}$ | $\frac{5}{6}$ |

Now, given the test result we need to distinguish two cases.

Recall $a = 0; b = 0; c = -2; d = 4$.

If we find (evidence for) a small market $R^S$

$$V_1^S \equiv \max\left[ p_1^S a + (1 - p_1^S)b,\ p_1^S c + (1 - p_1^S)d \right] = \max\left[0, -2 + 0\right] = 0.$$

If we find (evidence for ) a large market $R^L$

$$V_1^L \equiv \max\left[ p_1^L a + (1 - p_1^L)b, p_1^L c + (1 - p_1^L)d \right] = \max\left[0, \tfrac{1}{6} \cdot (-2) + \tfrac{5}{6} \cdot 4\right] = \tfrac{18}{6} = 3,$$

Prior to market research your probabilities to see evidence for a “small” or “large” market are

$$\Pr^S = p_0\varphi + \left(1 - p_0\right)\left(1 - \psi\right) = 0.4 \quad \text{and} \quad \Pr^L = p_0\left(1 - \varphi\right) + \left(1 - p_0\right)\psi = 0.6.$$

The expected payoff of performing a test is the weighted average of $V_1^S$ and $V_1^L$, where the weights are given by the probability of a test being positive or negative. Thus we have

$$V_1 = \Pr^S V_1^S + \Pr^L V_1^L = 0.4 \cdot 0 + 0.6 \cdot 3 = 1.8$$

The expected value of information (*VOI*) is defined as the difference between the expected value from an optimal decision on the use of *s* with and without additional evidence from market research

$$VOI \equiv V_1 - V_0 = 1.8 - 1 = 0.8$$

*References: A comprehensive textbook*


- Hirshleifer, Jack / Riley, John G. (1992) The Analytics of Uncertainty and Information. Cambridge.

# *Part c.: Games with incomplete information*

*References*

*Textbook*

- Jehle and Reny, sections 7.2.3–7.3.7 and chapter 8.
- (Kreps, David M. (1990) Microeconomic Theory. Prentice Hall.)

*Seminal works*


- Akerlof, George (1970) The Market for Lemons: Qualitative Uncertainty and the Market Mechanism. Quarterly Journal of Economics 84, 488-500.
- Stiglitz, Joseph E. (1974) Incentives and Risk Sharing in Sharecropping. Review of Economic Studies 41(2), 219-255.
- Spence, Michael (1973) Job Market Signalling. Quarterly Journal of Economics 87(3), 355-374.
- Rothschild, Michael / Stiglitz, Joseph E. (1976) Equilibrium in competitive insurance markets: an essay on the economics of imperfect information. Quarterly Journal of Economics 90(4), 629-649.

# *Games and Information: Concepts*
## *(for extensive form games)*

- *Perfect information game: Players, their strategy sets and payoffs are known for each stage of the game. At each stage and for each history of the game exactly one player has a non-trivial choice. This implies a simple tree structure of the game. Each information set is a singleton. It also implies perfect recall.*
- *Imperfect information: Players, their strategy sets and payoffs are known for each stage of the game, but not all aspects of past play. There exists at least one information set that is non-singleton. Examples are simultaneous move games.*
- *Incomplete information: Some information about other player, their payoffs or their strategies are missing.*
- *Common knowledge (of rationality) assumption (Robert Aumann): Players are rational, they know that they are rational and they know that they know that they are rational and so forth...*

## *A principal agent problem (Kreps 1990, chapter 16)*

A sales agent can work hard (high effort) or not (low effort). His disutility of effort is $a_h = 5$ and $a_l = 0$ , respectively. Consider the following utility function $u(w,a) = \sqrt{w} - a$. Let's first consider a benchmark case without uncertainty. Assume the agent brings home orders worth 70 if effort is low and 270 if effort is high. Furthermore the agent has a reservation utility of $\bar{u} = 9$.

What is the reservation wage of the agent?

For low effort: $\sqrt{w} - a_l \geq 9 \Rightarrow \sqrt{w} \geq 9 \Rightarrow w \geq 81$.

For high effort: $\sqrt{w} - a_h \geq 9 \Rightarrow \sqrt{w} \geq 14 \Rightarrow w \geq 196$.

What does the principal offer?

The contract offered pays the agent wage 196 + a cent if effort is high and 25 (say) otherwise.

- It is optimal for the agent to accept and work hard.
- Enforceability is not an issue as effort is assumed to be observable.

Now consider unobservable effort

The salesman (agent) brings home orders worth 400, 100 or 0 depending on luck and effort (see table):

| | **400** | **100** | **0** | **expected sales** |
|---|---|---|---|---|
| **high effort** | 0.6 | 0.3 | 0.1 | 270 |
| **low effort** | 0.1 | 0.3 | 0.6 | 70 |

The principal is assumed to be risk neutral. Effort is not observable, but the size of the sale is.

What does the principle believe if he sees a large sale?

Suppose his prior belief that the agent exerts high effort is 0.5. Then, seeing a large sale the belief will be adjusted to 6/7.

<u>Optimal contract offered to a risk neutral agent</u>

In this case assume

$u(w,a) = w - a;\ \bar{u} = 81; a_h = 25; a_l = 0.$

Hence the reservation wage for hard work is $81 + 25 = 106$.

Principal's expected profits are $270 - 106 = 164$.

For low efforts the principal would rather not hire the agent, since 81 > 70.

What contract should the principal offer?

The wage will be dependent of sale.

$$w = \begin{cases} -164 & \text{for no sale} \\ -64 & \text{for small sale} \\ 236 & \text{for large sale} \end{cases}$$

Agent exerts high effort, earns the reservation wage and bears all risk. Principal earns 164 for sure. You can check that it does not make sense for the agent to accept the contract and exert low effort.

Optimal contract offered to a risk averse agent

In this case assume as before

$u(w,a) = \sqrt{w} - a;\ \bar{u} = 9; a_h = 5; a_l = 0.$

**Observe the trade-off:**

- **The risk neutral principal should bear all the risk,**
- **But a riskless wage offers no incentives for effort.**

Denote, for convenience, the utility equivalent of $w$ by $x$, such that $x^2 = w$. The principles problem is to offer a contract

$$w = \begin{cases} x_0^2 & \text{for} \quad \text{no sale} \\ x_1^2 & \text{for} \quad \text{small sale} \\ x_2^2 & \text{for} \quad \text{large sale} \end{cases}$$

The principal maximises profits subject to

- The agent must be willing to accept the contract (participation constraint)
- The agent prefers high effort to low effort (incentive constraint)

Formally:

$$\max\ 270 - (0.1 \cdot x_0^2 + 0.3 \cdot x_1^2 + 0.6 \cdot x_2^2)$$

subject to

(PC) $9 \leq 0.1 \cdot x_0 + 0.3 \cdot x_1 + 0.6 \cdot x_2 - 5$

(IC) $0.6 \cdot x_0 + 0.3 \cdot x_1 + 0.1 \cdot x_2 \leq 0.1 \cdot x_0 + 0.3 \cdot x_1 + 0.6 \cdot x_2 - 5$

This is left to you as an exercise in constrained optimisation.

We find

$$w = \begin{cases} 5.43 & \text{for} \quad \text{no sale} \\ 14 & \text{for} \quad \text{small sale} \\ 15.43 & \text{for} \quad \text{large sale} \end{cases}$$

The expected wage is 204.5 and exceeds the reservation wage.
But notice that the agent bears some (but not all) risk and he is at his reservation utility level.
Expected profits are $270 - 204.5 = 65.5$.