Advanced Microeconomics(UEC 51806)

Classroom Exercises (Part 3)

Expected Utility Theory

1.
The Continuity axiom (Jehle and Reny, p.100) says that for any gamble $g \in \mathcal{G}$ there is some probability α such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$. Prove that α is unique for any g. (Hint use Monotonicity and Transitivity.)

Proof: We prove this by contradiction. Suppose $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$ and assume $g \sim (\beta \circ a_1, (1-\beta) \circ a_n)$ with $\beta \neq \alpha$. Let's assume $\beta > \alpha$.
Using Monotonicity we have
$(\beta \circ a_1, (1-\beta) \circ a_n) \succsim (\alpha \circ a_1, (1-\alpha) \circ a_n)$ since $\beta \geq \alpha$ and
$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim g$ by assumption and
$g \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ by assumption.
Then by transitivity $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ which violates Monotonicity since $\beta > \alpha$. So β must equal α.

2.
Show that if $u(w)$ represents a person's von Neumann-Morgenstern preferences, then $v(w) = \alpha u(w) + \beta$ with $\alpha > 0$ represents the same preferences.

Answer: We know that the ratio of utility differences is unique for a preference ordering. Suppose *a*, *b*, *c* are three lotteries with
$u(a) \succ u(b) \succ u(c)$. Then there is a unique $\gamma = \frac{u(a)-u(b)}{u(b)-u(c)}$. Now we only need to show that also $\frac{v(a)-v(b)}{v(b)-v(c)} = \gamma$ holds. To show this, simply substitute $v(w) = \alpha u(w) + \beta$ such that

$\frac{v(a)-v(b)}{v(b)-v(c)} = \frac{\alpha u(a)+\beta-(\alpha u(b)+\beta)}{\alpha u(b)+\beta-(\alpha u(c)+\beta)}$. Since we can cancel β and α from the latter term, we see that

$\frac{v(a)-v(b)}{v(b)-v(c)} = \frac{u(a)-u(b)}{u(b)-u(c)} = \gamma$.

3.
Which of the following (Bernoulli) utility functions have the DARA property, that is, the Arrow-Pratt measure of absolute risk aversion R_a is decreasing in wealth.

(i) $u(w) = \sqrt{w}$
(ii) $u(w) = \ln w$
(iii) $u(w) = -a(w^2 - 2w)$; $a > 0$

Answer:

$u(w)$	u'	u''	$R_a = -u''/u'$	DARA
$u(w) = \sqrt{w}$	$\frac{1}{2\sqrt{w}}$	$-\frac{1}{4}w^{-\frac{3}{2}}$	$\frac{1}{2}w^{-1}$	Yes
$u(w) = \ln w$;	$\frac{1}{w}$	$-\frac{1}{w^2}$	$\frac{1}{w}$	Yes
$u(w) = -a(w^2 - 2w)$; $a > 0$	$-a(2w-2)$	$-2a$	$-\frac{-2a}{-a(2w-2)} = \frac{1}{1-w}$	No

The relevant part of the quadratic utility function is constrained to $w \leq 1$.

4. Consider the following two lotteries:

$$L : \begin{cases} 200 \text{ with probability } 0.7 \\ 1 \text{ with probability } 0.3 \end{cases}$$

$$L' : \begin{cases} 2000 \text{ with probability } 0.1 \\ 1 \text{ with probability } 0.9 \end{cases}$$

Calculate their respective certainty equivalents c and c' for $u(w) = \ln w$.

Answer: We need to solve

$$0.7 \ln(200) + 0.3 \ln(1) = \ln(c)$$

$$\Rightarrow 0.7 \ln(200) = \ln(c)$$

$$\Rightarrow 200^{0.7} = c = 40.8$$

and

$$0.1 \ln(2000) + 0.9 \ln(1) = \ln(c')$$

$$\Rightarrow 0.1 \ln(2000) = \ln(c')$$

$$\Rightarrow 2000^{0.1} = c' = 2.14$$

5.

Show that if preferences are transitive and monotone the individual must prefer lottery L to L' if and only if $c > c'$.

ANSWER:

We have $c \circ 1 \sim L$ and $c' \circ 1 \sim L'$. WLG (without loss of generality) let $c > c'$. By monotonicity the larger prize is preferred, $c \circ 1 \succ c' \circ 1$ and by transitivity the claim follows. Note that this answer is general and does not depend on how L and L' are specified. Can you think of two utility functions where one gives $c > c'$ and the other the reverse?

6. A sports fan has subjective probability p that her favourite club "Ajax" will win the next match and probability $1-p$ that they will not win. Suppose she makes a bet x on Ajax such that if Ajax wins, she wins €x, but if Ajax does not win she loses €x. The fan's initial wealth is w_0.

(a) How much does she bet if she is risk neutral? How does her belief affect her bet?

Now suppose her Bernoulli utility function is $u(w)=\sqrt{w}$.

(b) Determine the fan's degree of absolute and relative risk aversion.

(c) How can you determine the fan's subjective odds $p/(1-p)$ by observing the size of her bet?

(d) Derive the optimal size of the bet x as a function of the subjective belief p.

(e) How much would she bet if $p=\frac{3}{5}$ and $w_0=13$?

Now suppose for Ajax's next game the fan has received a betting ticket from a friend as a gift. If Ajax wins, she receives x. (If Ajax does not win, her betting ticket has no value). This time $p=1/2$.

(f) For how much would the fan sell her betting ticket depending on x and w_0?

(g) Calculate the subjective money value of the gift when $x=3$ and $w_0=13$.

Answer

a) She bets nothing if $p<\frac{1}{2}$ and all her wealth if $p>\frac{1}{2}$.

b) The agent is risk averse $R_a=\frac{1}{2w}; R_r=\frac{1}{2}$.

c) A risk neutral player would bid any amount if $\frac{p}{1-p}>1$. A risk averse agent balances "trust" and risk aversion. Hence the agent maximizes.

$V=pu(w_0+x)+(1-p)u(w_0-x)$. From the first order conditions we obtain $\frac{p}{1-p}=\frac{\sqrt{w_0+x}}{\sqrt{w_0-x}}$.

d) Solving for x gives $x=\frac{w_0(2p-1)}{2p^2-2p+1}$. This is upward sloping, and positive for $p>\frac{1}{2}$.

e) Using the latter formula, we can use the numbers to obtain $x=5$.

f) Now we need to calculate the certainty equivalent value w_c of the value of the lottery ticket t when eventual losses are paid for

$u(w_{ce})=p\,u(w_0+x)+(1-p)u(w_0)$.

Using $p=1/2$ and the utility specification we get

$$\sqrt{w_0+t}=\tfrac{1}{2}(\sqrt{w_0+x}+\sqrt{w_0})$$
$$\Leftrightarrow t=\tfrac{1}{4}\left(x-2w_0+2\sqrt{w_0}\sqrt{x+w_0}\right).$$

g) Using the numbers we can calculate $t=2\sqrt{13}-\frac{23}{4}\approx 1.46$.

7. Consider the widget factory of Dixit and Pindyck (1994).
To be able to produce the firm must invest $I = 1600$. If it invests the firm can produce 1 item each year for ever. Assume zero production cost.
The current price is 200. The discount rate is 10%.
(a) Should the firm invest? What profits would it make.

Now consider uncertainty. Future prices may be up to 300 with probability q or down to 100 with probability $1-q$.

(b) What should the firm do? Invest, not invest, or wait?

(c) Would it pay for the firm to hire a market research institute that can make a perfect forecast of future prices? How much would the firm be willing to pay for that information.

Answer

a) The NPV of profits from the project is $V_{\text{certain future}} = -I + p\frac{r+1}{r} = -1600 + 200\frac{1.1}{0.1} = 600$.
b) We calculate the outcomes for all three options and compare.
$V_{\text{not invest}} = 0$.

$$V_{\text{invest}} = -I + p_0 + qp_h\frac{1}{r} + (1-q)p_l\frac{1}{r} = -1600 + 200 + 3000q + 1000(1-q)$$
$$= -400 + 2000q.$$

If you decide to wait you will learn the price and only invest in the next year if the price is high.

$$V_{\text{wait}} = -Iq\frac{1}{1+r} + qp_h\frac{1}{r} = -1455q + 3000q = 1545q.$$

Comparing the first two options we can conclude that it is better to invest if $q \geq \frac{1}{5}$.
Comparing the last two options it is better to invest immediately if

$$2000q - 400 \geq 1545q \Leftrightarrow q \geq \frac{400}{455} = 0.88.$$

Only if you are almost sure that price will be high you would invest immediately. Otherwise you wait. After waiting you would invest if prices are high but not invest if prices are low.

c) If you receive the information immediately the payoff is

$$V_{\text{full information}} = q(I + p_0 + p_h\frac{1}{r}) + (1-q)0 = q(-1600 + 200 + 3000) = 1600q$$

The value of information is

$$\text{VOI} = \begin{cases} \text{if } q \leq 0.88 & 1600q - 1545q = 55q \\ \text{if } q > 0.88 & 1600q - (2000q - 400) = 400 - 400q \end{cases}$$

The VOI function is triangular with its peak at 0.88.

Games with incomplete information

8.

The stag hunt game. Two hunters can hunt for stag or hare.

	stag	hare
stag	9, 9	0, 8
hare	8, 0	7, 7

Determine all Nash equilibria of the game.

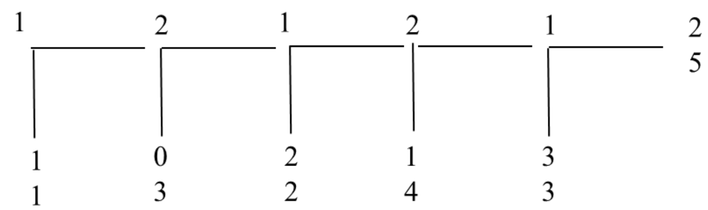
Answer: {stag, stag} and {hare, hare} are the pure strategy equilibria. There is a mixed strategy equilibrium if players choose "stag" with probability 7/8.

The mixed strategy equilibrium can found as follows. A player would randomise her choice if she is indifferent between "stag" and "hare". Consider her belief that the other plays stag is p. Then for the mixed strategy equilibrium it must hold that $9p + 0(1-p) = 8p + 7(1-p)$.

Hence, in the equilibrium we must have $p = \frac{7}{8}$. Check that if player 1 plays this mixed strategy, then player 2 is indifferent between playing "stag" and playing "hare".

9.

The centipede game. Players move alternatingly. At each node they have two options *down* or *across*.



Find the equilibrium

Answer: By backward induction the unique equilibrium is that player 1 moves down immediately and would play down at all nodes. Player 2 would also play down at all nodes. This equilibrium is subgame perfect.

10.

The market for lemons (Akerlof). Consider the following market for used cars. There are many sellers of used cars and even more buyers. Each seller has exactly one used car to sell and is characterised by the quality of the used car he wishes to sell. Let $\theta \in [0,1]$ index the quality of a used car and assume that θ is uniformly distributed on [0, 1]. If a seller of type θ sells his car (of quality θ) for a price p, his utility is $u_s(p,\theta)$. If he does not sell his car, then his utility is 0. Buyers of used cars receive utility $u_b = \theta - p$ if they buy a car of quality θ at price p and receive utility 0 if they do not purchase a car. There is asymmetric information regarding the quality of used cars. Sellers know the quality of the car they are selling, but buyers do not know its quality. Assume that there are not enough cars to supply all potential k buyers.

a) Argue that in a competitive equilibrium under asymmetric information, we must have $\mathrm{E}(\theta \mid p) = p$.

b) Show that if $u_s(p,\theta) = p - \frac{\theta}{2}$, then every $p \in [0, \frac{1}{2}]$ is an equilibrium price.

c) Find the equilibrium price when $u_s(p,\theta) = p - \sqrt{\theta}$. Describe the equilibrium in words. In particular, which cars are traded in equilibrium?

d) Find an equilibrium price when $u_s(p,\theta) = p - \theta^3$. How many equilibria are there in this case?

ANSWER

(a) In the competitive equilibrium demand for used cars must equal supply? There are two cases to consider. If $E(\theta \mid p) < p$, then the conditional average quality of cars on the market is below the price and buyers receive negative expected utility. So there is no demand and the market cannot clear.

On the other hand, if $E(\theta \mid p) > p$, then buyers receive positive expected utility and all potential buyers would like to buy but not all sellers would like to sell (those with high quality car will not sell as the price is less than average quality). So there is excess demand. The equilibrium outcome is driven by positive selection: a higher price increases the average quality of the cars available on the market. So there are potentially many equilibria solving the pricing condition $E(\theta \mid p) = p$. There might exist a high price equilibrium where sellers put their high quality cars on the market and buyers are willing to pay the high price. There might also exist a low price equilibrium where the sellers remove the best cars from the market and buyers are only willing to pay a low price.

(b) Fixing the price p, if $u_s(\theta \mid p) = p - \frac{\theta}{2}$, all sellers with quality $\theta \leq 2p$ will prefer selling their car to not selling. Thus the average quality of cars conditional on price p is given by $E(\theta \mid p) = \min(\frac{2p}{2}, \frac{1}{2}) = p$ which is also the competitive equilibrium outcome. Thus any price $p \in [0, \frac{1}{2}]$ is an equilibrium with only cars of quality $\theta \leq 2p$ traded, so that the average quality is just equal to the price.

(c) When $u_s(\theta \mid p) = p - \sqrt{\theta}$, only sellers with quality $\theta \leq p^2$ will prefer to sell. Thus the quality threshold $\theta^* = p^2$ which yields the conditional average quality of cars of $\frac{p^2}{2}$. Then $E(\theta \mid p) = \min(\frac{p^2}{2}, \frac{1}{2}) = p$ and $p = 0$ is the only possible solution.

(d) Similar to cases (b) and (c) we have $E(\theta \mid p) = \min(\frac{p^{1/3}}{2}, \frac{1}{2}) = p$. Such that we obtain $p = \sqrt{\frac{1}{8}}$. Another equilibrium is $p = 0$. Clearly there is market failure because high quality cars do not sell. Pareto improvements are not possible unless we change the game. One possibility is that sellers give guarantees.