Advanced Microeconomics: Game Theory

Lesson 5:
Games in Extensive Form (part 2)

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What You will learn

After studying Lesson 5, You

- should understand the notion of subgame perfect Nash equilibrium;
- should know how to perform the procedure of backward induction;
- should know the Theorem of Kuhn;
- should understand the notions of value and optimal strategy.

Let us reconsider the game with matches in Lesson 4: there is a pillow with 100 matches. They alternately remove $1, 3$ or 4 matches from it. (Player 1 begins.) The player who makes the last move wins.

We have shown that in this game player 2 has a winning strategy by analysing the game from the end to the beginning. In fact what there was done is what is called the procedure of backward induction . This procedure applies to each game in extensive form.

Procedure of backward induction

This procedure can be explained by means of the following video which deal with a concrete game in extensive form. Have a look to this video if You like.

`https://www.youtube.com/watch?v=GoeX3fNKghQ`.

So what one learns from this video is that in a game in extensive form there may be various Nash equilibria which can be divided in good (i.e. credible) ones and bad (i.e. incredible) ones.

I will explain these things now more systematically.

Subgame perfect Nash equilibria

Well, a credible Nash equilibrium also is called subgame perfect Nash equilibrium . It is a Nash equilibrium that remains a Nash equilibrium in each subgame. Here with subgame we mean the game obtained from the original game tree by starting at one of its decision nodes. (Of course, the original game starts at its initial decision node.)

Subgame perfection

In order to find the subgame perfect Nash equilibria the procedure of backward induction (as explained in the video and on the blackboard) is important.

This procedure (also referred to as ‘pruning the tree’) leads to a non-empty set of strategy profiles, so called backward induction strategy profiles .

Very important results

Here are three important results:

Theorem

(Kuhn.) Each backward induction strategy profile of a finite game in extensive form with perfect information is a Nash equilibrium.

However not each Nash equilibrium is a backward induction strategy profile (i.e. is subgame perfect):

Theorem

For every finite extensive form game with perfect information the set of backward induction strategy profiles coincides with the set of subgame perfect Nash equilibria.

The proofs of both these results are delicate. See text book (if You like).

Very important results (ctd.)

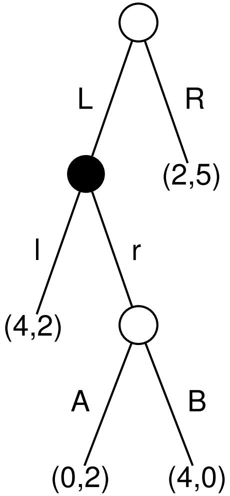
As the set of backward induction strategy profiles is not empty, the last result implies:

Theorem

Every finite extensive form game with perfect information has a subgame perfect Nash equilibria.

Example

Let us consider again the following example from Lesson 4:



Example (ctd.

We have seen that player 1 has 4 strategies: *LA*, *LB*, *RA* and *RB*. And that player 2 has 2 strategies: *l* and *r* and that normalisation leads to the bimatrix game

$$\begin{pmatrix} & l & r \\ LA & 4;2 & 0;2 \\ LB & 4;2 & 4;0 \\ RA & 2;5 & 2;5 \\ RB & 2;5 & 2;5 \end{pmatrix}.$$

Example (ctd.)

This bimatrix game has two Nash equilibria: $(LA, l), (LB, l)$.

Note that there are 3 subgames, one for each decision node.

Performing the procedure of backward induction we obtain: (LB, l) is a subgame perfect Nash equilibrium, (LA, l) is not a subgame perfect Nash equilibrium.

Value and optimal strategy

Consider an antagonistic game, i.e. a game in extensive form with two players where (at the end nodes) the sum of the payoffs is zero.

As we suppose that the game is finite and with perfect information it has a subgame perfect Nash equilibria. Each subgame perfect Nash equilibria is a Nash equilibrium of the normalisation (being a game in strategic form) of the extensive form game. We know that the payoff of player 1 is independent of the Nash equilibrium. This payoff, denoted, by v, is called the value of the game.

An optimal strategy for player 1(2) is a strategy $e_1(e_2)$ where (e_1, e_2) is any Nash equilibrium.

Value and optimal strategy (ctd.)

Also see Exercise 3 in Exercise set 4.

This exercise shows that for an antagonistic game with value v an optimal strategy for for a player 1 (2) is a strategy for this player that guarantees this player at least a payoff v ($-v$).

Hex revisited

We just have seen that Hex has a value and also already know that this value is not 0; so it is $+1$ (i.e. player 1 has a winning strategy) or -1 (i.e. player 2 has a winning strategy). We prove that for $m \times m$ Hex this value is $+1$. The clever proof that is presented here is a so-called ‘strategy stealing argument’. It is a proof by contradiction.

Hex revisited (ctd.)

So suppose player 2 has a winning strategy which we call S; we deal here with an $m \times m$ board. Now we make as follows a winning strategy for player 1; so we have a contradiction. Player 1 makes his first move at random. Thereafter he should pretend to be player 2, 'stealing' the second player's strategy S, and follow S. What, of course, we still have to show is that this is a legitimate way of playing the game. Well, if strategy S calls for him to move in the hexagon that he has chosen at random before (so for example for his first move), he should choose at random again. This will not interfere with the execution of S. Finally, note that the above way of playing for player 1 is at least as good as strategy S of player 2 since having an extra marked square on the board is never a disadvantage in Hex. Thus player 1 wins.