Advanced Microeconomics: Game Theory

Lesson 4:
Games in Extensive Form (part 1)

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What You will learn

After studying Lesson 4, You

- should know what we mean by a game in extensive form and know various notions for such a game;
- should be able to transform a games in extensive form into a game in strategic form by means of normalisation;
- should understand the idea of ‘solving a game from the end to the beginning’.

Game in extensive form

The setting in the this lesson always is a non-cooperative one with complete information and perfect information and no chance moves.

A formal definition of a game in extensive form is quite technical: see Definition 7.13 in the text book. I shall proceed here now less formally.

Game in extensive form (ctd.)

A game in extensive form can be represented by a game tree. In such a tree

- there are nodes (also called histories): end nodes, decision nodes and a unique initial node;
- there are a (directed) branches;
- there are payoffs at the end nodes;
- each non-initial node has exactly one predecessor.

We further always assume that the game is finite, i.e. that the number of nodes and branches is finite.

Game in extensive form (ctd.)

Outcome of a game in extensive form : a path through the game tree, or equivalently an end node of the game tree.

A primary purpose of game theory is to determine the outcomes of games according to a solution concept (e.g. Nash equilibrium).

Notion of strategy

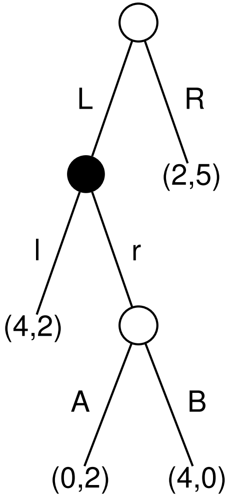
Given a game in extensive form, for a player the notion of completely elaborated plan of playing is defined. One may consider it as a piece of paper on which that player writes which move he will make if it is his turn.

A related notion is that of strategy of a player: this is specification at each decision node how to move.

Note that a strategy may be much more than a completely elaborated plan of playing: the player also has to specify his moves at nodes which never may be reached. This sounds strange. In Lesson 5 we shall see why this notion of strategy is very important in order to understand how games will be played.

Example

Here is an example of a game tree:



Example (ctd.)

In this game tree, the nodes where player 1 moves are presented by the $\circ$ symbol and those for player 2 by the $\bullet$ symbols.

The possible moves are denoted by the symbols L, R, l, r, A, B.

The payoffs are at given at the end nodes (for which no symbol is used).

Example (ctd.)

In this game

- Player 1 has 4 strategies: LA, LB, RA and RB.
- Player 2 has 2 strategies: l and r.
- Playing R is a completely elaborated plan of playing for player 1, but it is not a strategy. If You do not understand this, then look back to the definition of strategy!

This game will be dealt with further in a moment.

Normalisation

Out of a given a game in extensive form, one can make in a natural way a game in strategic form: just consider for each player all possible strategies and calculate the payoffs at each possible strategy profile.

This transformation of a game in extensive form into a game in strategic form is called normalisation .

As a game in strategic form is a game with imperfect information, normalisation destroys the perfect information.

Normalisation makes that all terminology and results for games in strategic form now also applies to games in extensive forms. In particular the notion of Nash equilibrium.

Example (ctd.)

Again consider the above game tree.
We have seen that player 1 has 4 strategies: *LA*, *LB*, *RA* and *RB*. And that player 2 has 2 strategies: *l* and *r*.
Normalisation leads to the bimatrix game

$$\begin{pmatrix} & l & r \\ LA & 4;2 & 0;2 \\ LB & 4;2 & 4;0 \\ RA & 2;5 & 2;5 \\ RB & 2;5 & 2;5 \end{pmatrix}.$$

Antagonistic game

An antagonistic game is a game in extensive form with two players where (at the end nodes) the sum of the payoffs is zero.

Many parlor games are antagonistic.

Let us now, as an appetizer for what follows, look to some antagonistic games.

Appetizer

Tic-tac-toe, Chess, (8×8) Checkers and Nim end with a winner (and a loser) or with a draw. We also already know that Hex cannot end in a draw and so each Hex game ends with a winner.

We shall see that we can make very strong predictions how rational intelligent players will play these games. The reason is, that as we shall see, that all these games game have a value which will be realized if each player plays an optimal strategy.

Appetizer (ctd.)

Here are the results for these games:

	Tic-t.-t.	Chess	Checkers	Hex	Nim
value	draw	not known	draw	1 wins	known
opt. strat.	known	not known	known	not known	known

The notions of value and optimal strategy will be defined formally in the next lesson.

Solving from the end to the beginning

As an introduction to the next lesson, we now consider the following game between two players. There is a pillow with 100 matches. They alternately remove $1, 3$ or 4 matches from it. (Player 1 begins.) The player who makes the last move wins. How this game will be played?

The idea to handle this question is to analyse the game “from the end to the beginning”:

Solving from the end to the beginning (ctd.)

The idea is to classify the positions of the game as winning or loosing. A winning position means that the player who has to move at this position can guarantee himself winning. And a loosing position means that the player who has to at this position can not guarantee himself winning (and looses if the opponent plays optimally).

If there are 0 matches left, then the player who has to play loses. So 0 is a losing position

If there is 1 match left, then the player who has to play wins. So 1 is a winning position.

If there are 2 matches left, then the player who has to play has to remove one match and a position which 1 match remains. As this is a winning position, 2 is a losing position.

Solving from the end to the beginning (ctd.)

If there are 3 matches left, then the player who has to play can remove these matches and then wins. So 3 is a winning position.

If there are 4 matches left, then the player who has to play can remove these matches and then wins. So 4 is a winning position.

If there are 5 matches left, then the player who has to play can remove 3 matches and a position which 2 matches remains. As this is a losing position, 5 is a winning position.

If there are 6 matches left, then the player who has to play can remove 4 matches and a position which 2 matches remains. As this is a losing position, 6 is a winning position.

Solving from the end to the beginning (ctd.)

If there are 7 matches left, then the player who has to play always will end up in a winning position. So 7 is a losing position.

And so on ... . Inspecting these numbers it is not so difficult to see that the loosing positions are $0, 2, 7, 9, 14, 16, 21, \ldots$, i.e. the numbers that have remainder 0 or 2 when divided by 7.

Because $100/7$ has remainder 2, 100 is a loosing position. Conclusion: player 2 always can win the game, implying that he has a winning strategy.