Advanced Microeconomics: Game Theory

Lesson 3:
Games in Strategic Form (part 2)

P. v. Mouche

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What You will learn

After studying Lesson 3, You

- should understand, for games in strategic form, the introduced game theoretic vocabulary formed by the fundamental notions.
- should know how to make predictions by using solution concepts.
- should be able to deal with mixed strategies for bimatrix games.

Main mathematical types

Concerning the mathematical structure of games one can distinguish between three types of games:

- Games in strategic form.
- Games in extensive form. (We deal with in next two lessons.)
- Games in characteristic function form. (We will not deal with as they belong to cooperative game theory.)

Game in strategic form

Definition

Game in strategic form , specified by

- n players : $1, \ldots, n$.
- for each player i a strategy set (or action set) X_i. Let $\mathbf{X} := X_1 \times \cdots \times X_n$: set of strategy profiles .
- for each player i payoff function $f_i : \mathbf{X} \to \mathbb{R}$.

The elements of X_i are referred to as strategies (of player i) and thos of $\mathbf{X}$ as strategy profiles .

Interpretation: players choose simultaneously and independently a strategy.

Game in strategic form (ctd.)

A game in strategic form is called finite if each strategy set X_i is finite.

Of course, in the case of two players a finite game in strategic form can be represented as a bimatrix game. For example if each player has two strategies, say $X_1 = X_2 = \{1, 2\}$, then the bimatrix game is

$$\begin{pmatrix} f_1(1,1); f_2(1,1) & f_1(1,2); f_2(1,2) \\ f_1(2,1); f_2(2,1) & f_1(2,2); f_2(2,2) \end{pmatrix}.$$

Besides the choices in red on the slides in Lesson 1 concerning real-world-types, we further assume complete information, static game and for the moment no chance moves.

Please note that a game in strategic form (and in particular a bimatrix game) is a game with imperfect information as the moves are simultaneously.

Normalisation

Many games do not have the structure of a game in strategic form but that of a game in extensive form. However, as we shall see in Lesson 4 such games can be represented in a natural way as a game in strategic form.

This makes that in order to understand games in extensive form, it is important to understand games in strategic form.

Fundamental notions

The notions of dominant strategy, strictly dominant strategy, strictly dominant Nash equilibrium, Nash equilibrium, weakly Pareto efficient strategy profile, strongly Pareto efficient strategy profile, fully cooperative strategy profile, prisoner's dilemma), for a bimatrix game in the previous lesson, also make sense for an arbitrary game in strategic form. Their definition is exactly the same.

Please review these notions now!

Some additional fundamental notions will now be introduced.

Fundamental notions (ctd.)

- Conditional payoff function $f_i^{(\mathbf{z})}$ of player i: f_i as a function of the strategy x_i of player i for fixed strategy profile $\mathbf{z}$ of the opponents.
- Best response correspondence R_i of player i: assigns to each strategy profile $\mathbf{z}$ of the opponents of player i the set of maximisers $R_i(\mathbf{z})$ of $f_i^{(\mathbf{z})}$.
- Strongly (or strictly) dominated strategy of player i: a strategy x_i of a player for which there exists another strategy y_i of that player that independently of the strategies of the other players always gives a higher payoff than x_i.

Solution concepts

The aim of game theory is to understand/predict how games will be played. Here so-called solution concepts play a role. For games in strategic form the following one are important.

- Strictly dominant Nash equilibrium : strategy profile where each player has a strictly dominant strategy.
- Nash equilibrium : strategy profile such that no player wants to change his strategy in that profile.

We already are familiar with these notions.

More formally: a strategy profile $(x_1, \ldots, x_n)$ is a Nash equilibrium if and only if for every player i and $y_i \in X_i$

$$f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \leq f_i(x_1, \ldots, x_n).$$

Best response correspondences

The following fundamental relation between Nash equilibria and the best response correspondences hold:

A strategy profile $(x_1, \ldots, x_n)$ is a Nash equilibrium if and only of for every player i

$$x_i \in R_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

i.e. if no player regrets his choice.

Here is a new one:

- Procedure of iterative (simultaneous) elimination of strongly dominated strategies .
- Strategy profile that ‘survives’ this procedure .
- If there is a unique strategy profile that survives the above procedure this strategy profile is called the iteratively not strongly dominated equilibrium .

Please see the text book for a (formal) definition of these notions. Here, i shall explain them with some examples.

Solution concepts (ctd.)

Theorem

1. *Each strictly dominant N.e. is an it. not strongly dom. e.*

And if the game is finite:

2. *An iteratively not strongly dominated equilibrium is a unique Nash equilibrium.*
3. *Each Nash equilibrium is an iteratively not strongly dominated strategy profile. (So each Nash equilibrium survives the procedure.)*

Proof.

1. Already in first steps of procedure all strategies are removed with the exception of strictly dominant ones.
2, 3. One verifies that in each step of the procedure the set of Nash equilibria remains the same. (See the text book.) □

Examples

1. Determine the best response correspondences, the strictly dominant Nash equilibria, the iteratively not strongly dominated equilibria and the Nash equilibria of the game

$$\begin{pmatrix} 2;4 & 1;4 & 4;3 & 3;0 \\ 1;1 & 1;2 & 5;2 & 6;1 \\ 1;2 & 0;5 & 3;4 & 7;3 \\ 0;6 & 0;4 & 3;4 & 1;5 \end{pmatrix}.$$

Examples (ctd.)

Answer: $R_1(1) = \{1\}, R_1(2) = \{1, 2\}, R_1(3) = \{2\}, R_1(4) = \{3\}, R_2(1) = \{1, 2\}, R_2(2) = \{2, 3\}, R_2(3) = \{2\}, R_2(4) = \{1\}$. No strictly dominant Nash equilibrium. The procedure gives $\begin{pmatrix} 2;4 & 1;4 & 4;3 \\ 1;1 & 1;2 & 5;2 \end{pmatrix}$. Thus the game does not have an iteratively not strongly dominated equilibrium. Nash equilibria: $(1, 1), (1, 2), (2, 2)$ and $(2, 3)$.

Examples (ctd.)

2. Determine the strictly dominant equilibria, the iteratively not strongly dominated equilibria and the Nash equilibria of the game
$$\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}$$
Answer: No player has as strictly dominant strategy, thus the game does not have a strictly dominant Nash equilibrium. The procedure of iterative elimination of strongly dominated strategies gives the bimatrix $(5;2)$. Thus the game has an iteratively not strongly dominated equilibrium: $(3,3)$. The game has one Nash equilibrium: $(3,3)$.

Best response dynamics

Consider a game in strategic form with 2 players. Let $R_1 : X_2 \multimap X_1$ be the best response correspondence of player 1 and $R_2 : X_1 \multimap X_2$ be the best response correspondence of player 2.

Best-response dynamics concerns a process where players, one after the other, react: start at a strategy profile $(x_1^{(1)}, x_2^{(1)})$. Suppose player 1 reacts first. He replaces his strategy $x_1^{(1)}$ by $x_1^{(2)} \in R_1(x_2^{(1)})$. This leads to strategy profile $(x_1^{(2)}, x_2^{(1)})$. Next player 2 replaces his strategy $x_2^{(1)}$ by $x_2^{(2)} \in R_2(x_1^{(2)})$. This leads to strategy profile $(x_1^{(2)}, x_2^{(2)})$.

And so on. If this process stops, then the end strategy profile is a Nash equilibrium. (But, may be it does not stop ...).

Nash equilibria in the continuous case

So in this lesson we also deal with games, like the Cournot oligopoly, where each player has infinitely many strategies. For such games one needs calculus (may be even analysis) in order to determine various fundamental objects, like Nash equilibria.

Remember that in a Nash equilibrium no player regrets his choice. More formally, in the case of two players: a strategy profile (e_1, e_2) is a Nash equilibrium if e_1 maximises $f_1(x_1, e_2)$ as a function of x_1 and e_2 maximises $f_2(e_1, x_2)$ as a function of x_2.

Nash equilibria in the continuous case (ctd.)

Under conditions where one can find maxima by putting derivatives to zero (in economics such conditions are quite usual) one can find the Nash equilibrium by solving the two equations

$$\frac{\partial f_1}{\partial x_1} = 0 \text{ and } \frac{\partial f_2}{\partial x_2} = 0$$

in the two unknowns x_1, x_2.

More generally: Nash equilibria (often) are the solutions of the n equations

$$\frac{\partial f_i}{\partial x_i} = 0 \ (i = 1, \ldots, n)$$

in $x_1, \ldots, x_n$.

This is in particular the case where conditional payoff functions are differentiable and concave.

Mixed strategies

Some games do not have a Nash equilibrium. However such a game may have a Nash equilibrium, if one plays the strategies with probabilities. Such a strategy is called a mixed strategy; so here now we allow for chance moves.

- Mixed strategy of player i: probability density over his strategy set X_i.

If only one strategy has a positive probability of being selected, the player is said to use a pure strategy .

Mixed strategies (ctd.)

With mixed strategies, payoffs have the interpretation of expected payoffs. As the mixed strategy variant of a game in strategic form again is a game in strategic form, all above introduced fundamental notions, as that of Nash equilibrium, also make sense in the context of mixed strategies.

An important result is: each Nash equilibrium is a Nash equilibrium in mixed strategies. (See text book for formal proof.)

Bimatrix game with mixed strategies

Consider a 2×2 bimatrix game

$$(A; B)$$

(A concerns the part of the bimatrix for the row player and B the part for the column player.)

Strategies: $(p, 1-p)$ for player 1 and $(q, 1-q)$ for player B. This means player 1 plays row 1 with probability p (and row 2 with probability $1-p$). And player 2 plays column 1 with probability q (and column 2 with probability $1-q$).

Expected payoff functions:

$$\bar{f}_1(p, q) = (p, 1-p) * A * \begin{pmatrix} q \\ 1-q \end{pmatrix},$$

$$\bar{f}_2(p, q) = (p, 1-p) * B * \begin{pmatrix} q \\ 1-q \end{pmatrix}.$$

Example

Determine the Nash equilibria in pure strategies and the Nash equilibria in mixed strategies for

$$\begin{pmatrix} 0;0 & 1;-1 \\ 2;-2 & -1;1 \end{pmatrix}.$$

Answer: No Nash equilibria in pure strategies.

$$\bar{f}_1(p;q) = (p, 1-p) * A * \begin{pmatrix} q \\ 1-q \end{pmatrix} = \cdots = (-4q+2)p+3q-1,$$

$$\bar{f}_2(p;q) = (p, 1-p) * B * \begin{pmatrix} q \\ 1-q \end{pmatrix} = \cdots = (4p-3)q+1-2p.$$

Example (ctd.)

Solving $\frac{\partial \bar{f}_1}{\partial p} = -4q + 2 = 0$ and $\frac{\partial \bar{f}_2}{\partial q} = 4p - 3 = 0$ (for the Nash equilibria in mixed strategies which are not pure Nash equilibria) gives the Nash equilibrium in mixed strategies

$$p = 3/4, q = 1/2.$$

As there is no Nash equilibrium in pure strategies, the conclusion is that $p = 3/4,\ q = 1/2$ is the unique Nash equilibrium in mixed strategies.

Example

Determine the Nash equilibria in pure strategies and the Nash equilibria in mixed strategies for $\begin{pmatrix} -1;1 & 1;-1 \\ 1;-1 & -1;1 \end{pmatrix}$.

Answer: No Nash equilibria in pure strategies. Nash equilibrium in mixed strategies: $p = q = 1/2$.

Existence of Nash equilibria

Theorem

(Nikaido-Isoda.) Each game in strategic form where

1. *each strategy set is a convex compact subset of some* $\mathbb{R}^n$*,*
2. *each payoff function is continuous,*
3. *each conditional payoff function is quasi-concave,*

has a Nash equilibrium.

Proof.

This is a deep theoretical result. A proof can be based on Brouwer's fixed point theorem. See text book for the proof of a simpler case (Theorem 7.2., i.e. the next theorem). □

Theorem of Nash

Theorem

Each bi-matrix-game has a Nash equilibrium in mixed strategies.

Proof.

Apply the Nikaido-Isoda result. □

Zero-sum game with two players

Consider a zero-sum game with two players.

Theorem

(Cfr. with Exercise 7.7 in the text book.) If (a_1, a_2) *and* (b_1, b_2) *are Nash equilibria, then* $f_1(a_1, a_2) = f_1(b_1, b_2)$ *and* $f_2(a_1, a_2) = f_2(b_1, b_2)$.

Proof.

As (a_1, a_2) and (b_1, b_2) are Nash equilibria, we have $f_1(a_1, a_2) \geq f_1(b_1, a_2)$ and $f_2(b_1, b_2) \geq f_2(b_1, a_2)$. This implies $f_1(a_1, a_2) \geq f_1(b_1, a_2) = -f_2(b_1, a_2) \geq -f_2(b_1, b_2) = f_1(b_1, b_2)$.
In the same way $f_1(b_1, b_2) \geq f_1(a_1, a_2)$.
Therefore $f_1(a_1, a_2) = f_1(b_1, b_2)$ and thus $f_2(a_1, a_2) = f_2(b_1, b_2)$. □

Thus the payoff of player 1 is the same at all Nash equilibria.