Advanced Microeconomics: Game Theory

Lesson 1: Motivation and Outlook

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Welcome

Welcome to the second part of Advanced Microeconomics!

I’ll teach this part, dealing with game theory, in six Lessons.

Lesson 1: Motivation and Outlook.
Lesson 2: Games in Strategic Form (part 1).
Lesson 3: Games in Strategic Form (part 2).
Lesson 4: Games in Extensive Form (part 1).
Lesson 5: Games in Extensive Form (part 2).
Lesson 6: Congestion Games.

What You will learn

After studying this lesson, i.e. Lesson 1, You

- should be able to explain what game theory is about;
- should be familiar with the specific games dealt with in this lesson;
- should know which real-world types of games one distinguishes.

What is game theory?

Traditional game theory deals with mathematical models of conflict and cooperation in the real world between at least two rational intelligent players.

- Player: humans, organisations, nations, animals, computers,...
- Situations with one player are studied by the classical optimisation theory.
- ‘Traditional’ because of rationality assumption.

Nature of game theory

- Applications.
  - Economics: Nobel prices in 1994 for Nash, Harsanyi and Selten, in 2005 for Aumann and in 2007 for Meyerson and Maskin.
  - Sociology, psychology, antropology, politocology.
  - Military strategy.
  - Biology (evolutionary game theory).
  - Design of computer games and robots.
- Game theory provides a language that is very appropriate for conceptual thinking.
- Many game theoretical concepts can be understood without advanced mathematics.
- Aim of game theory is to understand/predict how games will be played.

Players

In general we shall denote the players by numbers. And in the case of n players by $1, \ldots, n$.

Further on, when dealing with theory, we often deal for simplicity with two players: player 1 and player 2, or white and black, ... (In practice, for parlor games, a device like a die may decide who is which player.)

Outcomes and payoffs

- A game can have different outcomes, i.e. ways the game can be played. Each outcome has its own payoffs for every player.
- Nature of payoff: money, honour, activity, nothing at all, utility, real number, ...
- Interpretation of payoff: ‘satisfaction’ at end of game.
- In many parlor games with two players, the payoff of a player can be represented as winning, draw or loosing.
- In general it does not make sense to speak about ‘winners’ and ‘losers’ (and/or ‘draw’). It does, however, in various parlour games, like Chess, Tic-Tac-Toe, Stone-Paper-Scissors (and Football).

Rationality and intelligence

- Because there is more than one player, especially rationality becomes a problematic notion. Here is a simple try: a player is rational if he has well-defined preferences concerning the outcomes of the game.
- Intelligence also is a not so easy notion. It presupposes an intelligent player and refers to the (rational) goal of that player. Intelligence has to do with the way the goal is approached.
- So rationality' and 'intelligence' are different concepts and the intelligence notion presupposes which type of rationality we are speaking about.
- In many games rationality is not a big assumption.

Making predictions

Assuming for the moment that You know (as should be, as this is an advanced course!) what a bimatrix game is, we can now introduce some notions that are useful for making predictions about reasonable outcomes of such a game. But first a motivating example:

$$\begin{pmatrix} -1; -1 & -3; 0 \\ 0; -3 & -2; -2 \end{pmatrix}.$$

What would You as player 1 play in this game: row 1 or row 2? May be Your answer (as quite often happens) is row 2 (as this is for You the best independent what Your opponent plays). And Your opponent may answer column 2 because of the same reason. Then the result will be a payoff of -2 for You both. However, playing row 1 and column 1 would be better for both of You.

Making predictions (ctd.)

This game is the classical prisoners' dilemma game (of A. Tucker) where the payoffs correspond to years in prison.

Do not worry (too much): Lesson 2 will pick up again the notion of bimatrix game.

Some concrete games

Now we shall consider various concrete games. These games will be used in order to illustrate the abstract theory that we develop in the next lessons. These games concern:

Parlor games:

- Tic-Tac-Toe.
- Hex.
- Nim.

Economic games:

- Cournot Oligopoly.
- Hotelling Game.
- Congestion Game.

Some concrete games (ctd.)

You can find much more concrete games on
`http://pvmouche.deds.nl/advmicro-2023-2024.html`
under ‘Additional’ of my part of the course.

Parlor games have strict rules. But economic games are game theoretic models (with strict rules) of real-world economic situations where rules are not strict.

In the above parlor games (ordinary) rationality means: winning is better than draw and draw is better than loosing.

Tic-Tac-Toe

- Tic-Tac-Toe is a very well-known game.
- (For example) here You can play this game online: `https://playtictactoe.org/`.
- The game has many outcomes: in fact, as can be shown, 255168 ones. However, there are only the following three types of outcomes: player 1 wins, draw, player 1 loses.
- Example of a play of this game:

Tic-Tac-Toe (ctd.)

X		

X		
		O

X	X	
		O

X	X	O
		O

X	X	O
X		O

X	X	O
		O
X		O

So: player 2 is the winner.

Tic-Tac-Toe (ctd.)

Question: is player 1 intelligent?
Answer: We do not know. However, if player 1 is rational, then he is not intelligent.

Question: is player 1 rational?
Answer: We do not know. However, if player 1 is intelligent, then he is not rational.

Hex

- Hex is a two player board game, played on a board consisting of $m \times n$ hexagons.
- Please see `http://www.lutanho.net/play/hex.html` and play this game several times.
- The game at the above web page has an 11×11 board.
- Hex was invented independently by Piet Hein (1942) and John Nash (1948).

Hex (ctd.)

The Hex game has very interesting properties:

- Hex can not end in a draw. ('Equivalent' with Brouwer's fixed point theorem in two dimensions, being a deep mathematical result.)
- We shall prove later after having developed some theory that player 1 always can win the $m \times m$ Hex game.
- If You can give a winning strategy for Hex for every size of the board, then You solved one of the six '1-million-dollar problems'. (Have a look to `https://en.wikipedia.org/wiki/Millennium_Prize_P` if You like.)

Cournot Oligopoly

A special topic in economics is industrial organization. The modern theory of industrial organization heavily relies on game theory; various market forms are considered. Here we consider the market form of Cournot Oligopoly . The Cournot Oligopoly is one of the oldest economic games.

A Cournot Oligopoly concerns firms in a competitive setting. There are various variants. We consider here the homogeneous duopoly: ‘duopoly’ concerns the assumption of two firms and ‘homogeneous’ that the firms sell the same article.

Cournot Oligopoly (ctd.)

The model is as follows: the firms, 1 and 2, simultaneously and independently supply an amount of the article to the market and then can sell it for a price depending on the total amount. With x_i the amount for firm i, the total amount is $X = x_1 + x_2$ and the price is $p(X)$. The function p is called price function (or inverse demand function). With c_i the cost function of firm i the profit of firm i, being revenue minus costs, is

$$\pi_i(x_1, x_2) = p(x_1 + x_2)x_i - c_i(x_i).$$

The function π_i is called profit function of firm i.

Hotelling Game

The (Discrete) Hotelling Game depends on a parameter m being a positive integer and a parameter w with $0 < w \leq 1$. Consider the $m+1$ points of $H := \{0, 1, \ldots, m\}$ on the real line, to be referred to as *vertices*.

0 1 2 3 4 5 $\cdots$ m

Two players simultaneously and independently choose a vertex. If player 1 (2) chooses vertex x_1 (x_2), then the hinterland of a player is the set of vertices that is the closest to his choice; a vertex that has equal distance to both players, a so-called ‘shared vertex’, belongs to both hinterlands. d In order to define

the payoffs resulting from the choices x_1 and x_2, it is good to first consider the case $w = 1$ (often referred to as inelastic demand).

Hotelling Game (ctd.)

So suppose $w = 1$. Then the payoff $f_i(x_1, x_2)$ of player i is the number of non-shared vertices in his hinterland and half times the number of shared vertices in his hinterland.

Now consider the general case where $0 < w \leq 1$; the case where $0 < w < 1$ is called 'elastic demand'. The payoff $f_i(x_1, x_2)$ is calculated as follows: a non-shared vertex in the hinterland of player i at distance d to x_i contributes w^d to his payoff and a shared vertex contributes $w^d/2$.

(The assignment for the second part of the course will deal with the Hotelling Game.)

Hotelling Game (ctd.)

This abstract definition allows for various interpretations.

For example (roughly stated): the real line part denotes locations, the vertices are consumers, the players are sellers and the payoffs are profits.

Or: the real line part denotes political opinions, the vertices are voters, the players are political parties and the payoffs are votes.
This game is another example of an economics game: in fact a discrete variant of the original Hotelling Game.

Hotelling Game; $m = 7$ and $w = 1$

Action profile (5,2) :

Payoffs:

$1 + 1 + 1 + 1 = 4$

$1 + 1 + 1 + 1 = 4$

Action profile (0,3) :

Payoffs

$1 + 1 = 2$

$1 + 1 + 1 + 1 + 1 + 1 = 6$

Hotelling Game; $m = 7$ and $w = 1$ (ctd.)

Action profile (2,6) :

Payoffs:

$1 + 1 + 1 + 1 + \frac{1}{2} = 4\frac{1}{2}$

$\frac{1}{2} + 1 + 1 + 1 = 3\frac{1}{2}$

Action profile (3,3) :

Payoffs:

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4$

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4$

Hotelling Game (ctd.)

Now consider the general case where $0 < w \leq 1$; the case where $0 < w < 1$ is called 'elastic demand'. The payoff $f_i(x_1, x_2)$ is calculated as follows: a non-shared vertex in the hinterland of player i at distance d to x_i contributes w^d to his payoff and a shared vertex contributes $w^d/2$.

Hotelling Game; $m = 5$ and $w = 1/4$

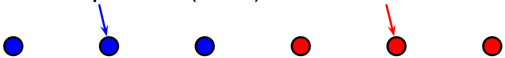
Action profile (1,3) :

Payoffs:

$\frac{1}{4} + 1 + \frac{1}{8} = 1\frac{3}{8}$

$\frac{1}{8} + 1 + \frac{1}{4} + \frac{1}{16} = 1\frac{7}{16}$

Action profile (1,4) :



Payoffs:

$\frac{1}{4} + 1 + \frac{1}{4} = 1\frac{1}{2}$

$\frac{1}{4} + 1 + \frac{1}{4} = 1\frac{1}{2}$

Hotelling Game; $m = 7$ and $w = 1/2$

Action profile (2,6) :

Payoffs:

$\frac{1}{4} + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{8} = \frac{19}{8}$

$\frac{1}{8} + \frac{1}{2} + 1 + \frac{1}{2} = \frac{17}{8}$

Nim

Nim is the following game. A certain number of piles consisting of a certain number of matches is put together. Both players take turns. Player 1 starts. Each turn a player must remove at least one match from a pile. The player that takes the last matche(s) wins. So there are infinitely many Nim games possible.

Consider Nim $(2, 3, 5, 5, 3, 2)$, i.e. there is one pillow with 2 matches, one with 3, ..., and one with 2. Play this game with an opponent. Be sure that You see how You can win this game if You are player 2.

Also play Nim $(5, 7, 6, 4, 1, 3, 9)$.

Congestion game

Will be introduced in Lesson 6.

Real-world types

In order to set up a theory for games one has to specify how the specific game that one considers relates to the real-word. (In red what we will assume always/sometimes later when we develop the theory.)

- all players are rational – players may be not rational
- all players are intelligent – players may be not intelligent
- binding agreements – no binding agreements
- chance moves – no chance moves
- communication – no communication
- static game – dynamic game
- transferable payoffs – no transferable payoffs

Real-world types (ctd.)

- interconnected games – isolated games
- perfect information – imperfect information
- complete information – incomplete information
- perfect recall – no perfect recall

The choices we made in red are very appropriate for dealing with non-cooperative game theory. Cooperative game theory (dealing with binding agreements) will not be dealt with in this course.

Below we briefly reconsider some of these notions.

Incomplete information vs imperfect information

One can say:

incomplete information refers to the amount of information the players have about the game, while imperfect information refers to the amount they have on others' and their own previous moves (and on previous chance moves).

Perfect information

- A player has perfect information if he knows at each moment when it is his turn to move how the game was played untill that moment.
- A player has imperfect information if he does not have perfect information.
- A game is with (im)perfect information if (not) all players have perfect information.
- Chance moves are compatible with perfect information.
- Examples of games with perfect information: Tic-tac-toe, Chess, ...
  Examples of games with imperfect information: Poker, Monopoly (because of the cards, not because of the die).

Complete information

- A player has complete information if he knows all payoff functions.
- A player has incomplete information if he does not have complete information.
- A game is with (in)complete information if (not) all players have complete information.
- Examples of games with complete information: Tic-Tac-Toe, Chess, Poker, Monopoly, ...
  Examples of games with incomplete information: Auctions, Oligopoly models where firms only know the own cost functions, ...

Common knowledge

Also common knowledge plays a role in game theory.

Something is common knowledge if everybody knows it and in addition that everybody knows that everybody knows it and in addition that everybody knows that everybody knows that everybody knows it and ...

Common knowledge (ctd.)

A group of dwarfs with red and green caps are sitting in a circle around their king who has a bell. In this group it is common knowledge that every body is intelligent. They do not communicate with each other and each dwarf can only see the color of the caps of the others, but does not know the color of the own cap. The king says: ”Here is at least one dwarf with a red cap.”. Next he says: “I will ring the bell several times. Those who know their cap color should stand up when i ring the bell.”. Then the king does what he announced.

Common knowledge (ctd.)

The spectacular thing is that there is a moment where a dwarf stands up. Even, when there are M dwarfs with red caps that all these dwarfs simultaneously stand up when the king rings the bell for the M-th time.

Do not worry if You do not see why this claim this true. (Try to understand that the claim is true if there are one dwarf with a red cap and two with a green cap; and if f there are two dwarfs with a red cap and three with a green cap.) Dealing with such things is quite advanced and as it turns out important for the fundamental basis of game theory. However, it is too advanced for our (relatively simple) Advanced Microeconomics course. But if You like to know why the claim is true, then please have a look at

`https://en.wikipedia.org/wiki/Common_knowledge_(logic)`

where a similar situation is dealt with.

Next lessons

The time has come to start with developing theory. This will happen in Lessons 2, 3, 4 and 5.

Among other things we shall: show that Tic-Tac-Toe and Hex and various other (parlor) games have a so-called value, in particular prove some claims made about Hex. Such games belong to a special domain of game theory: combinatoric game theory.

Of course, we also will explain how to make predictions of the outcomes of games without a value, like the Hotelling Game, Cournot Oligopoly, Congestion Game and various other economic games.

The most important notion will be that of Nash equilibrium.