Advanced Microeconomics: Game Theory

Lesson 2:
Games in Strategic Form (part 1)

P. v. Mouche

2022–2023, period 1

What You will learn

This is an advanced course and therefore we expect that You already know various things. However, for Your convenience, i shall in this lesson, Lesson 2, start from scratch. If You master the prerequisites for the Advanced Micro course, then probably You will not see many new things in this lesson.

Lesson 2 is devoted to fundamental game theoretic notions in the context of a bimatrix game (being a special case of a game in strategic form). In a next lesson i will pick up what is presented here, will generalise and further elaborate on it.

What You will learn (ctd.)

After studying Lesson 2, You should be able to determine for a bimatrix game

- the dominant strategies;
- the strictly dominant strategies;
- the strictly dominant equilibria;
- the Nash equilibria;
- the strongly Pareto efficient strategy profiles;
- the weakly Pareto efficient strategy profiles;
- the fully cooperative strategy profiles;
- whether the game is a prisoner’s dilemma.

These notions are very fundamental. In Lesson 3 we shall see additional notions.

What You will learn (ctd.)

We start our theory with the notion of bimatrix game and introduce the above already mentioned very fundamental notions for such a game.

After a training with such notions, we apply them (if possible) to the concrete games of Lesson 1.

Bimatrix game

So what is a bimatrix game ? Well, bimatrix games concern the most simple type of so-called games in strategic form dealing with two players; player 1 and player 2. The game is represented by a so-called bimatrix (which explains its name). For example:

$$\begin{pmatrix} 3;3 & 2;2 \\ 7;-1 & -3;1 \\ 1;2 & 12;-9 \end{pmatrix}.$$

- This is a 3×2-bimatrix game, i.e. it has 3 rows and 2 columns.

Bimatrix game (ctd.)

- Player 1 chooses a row: row 1, row 2 or row 3, meaning that player 1 has 3 strategies.
  Player 2 chooses a column: column 1 or column 2, meaning that player 2 has 2 strategies.
  These choices are made simultaneously and independently.
- In each of the cells of the bimatrix there is a pair of numbers, separated by a semicolon. These numbers represent the payoffs; the first number concerns player 1 and the second player 2.
- For example: at the strategy profile $(3, 2)$, i.e. row 3 and column 2, player 1 has payoff 12 and player 2 has payoff -9.

Many games can be represented in a natural way as a bimatrix game. For example stone-paper-scissors:

$$\begin{pmatrix} 0;0 & -1;1 & 1;-1 \\ 1;-1 & 0;0 & -1;1 \\ -1;1 & 1;-1 & 0;0 \end{pmatrix}$$

Indeed: first strategy is stone, second paper and third scissors. If players make the same choice, then it is draw: payoffs 0 for both. If players make a different choice, then there is a winner with payoff 1 and a looser with payoff -1.

Fundamental notions

Knowing what a bimatrix game is, we can now introduce some notions that are useful for making predictions about how such a game can be played. I focus here just on introducing these notions.

- strategy profile : for each player a strategy. For example the strategy profile $(3, 1)$ means: player 1 chooses strategy (i.e. row) 3 and player 2 chooses strategy 1.
- Strictly dominant strategy of a player: the best strategy of that player independently of strategies of the other players.
- (Weakly) dominant strategy of a player: a best strategy of that player independently of strategies of the other players.
- Strictly dominant equilibrium : strategy profile in which each strategy is strictly dominant.
- Nash equilibrium : strategy profile with the property that no player regrets his choice.

Fundamental notions (ctd.)

- A strategy profile **b** is an unanimous Pareto improvement of a strategy profile **a** if each player has in **b** a greater payoff than in **a**.
- A strategy profile **b** is a Pareto improvement of a strategy profile **a** if at least one player has in **b** a greater payoff than in **a** and no player has in **b** a smaller payoff than in **a**.
- A strategy profile **x** is weakly Pareto-efficient if there does not exist an unanimous Pareto-improvement of **x**.
- A strategy profile **x** is weakly Pareto-inefficient if there exists an unanimous Pareto-improvement of **x**.
- A strategy profile **x** is (strongly) Pareto-efficient if there does not exist a Pareto-improvement of **x**.
- A strategy profile **x** is (strongly) Pareto-inefficient if there exists a Pareto-improvement of **x**.

Fundamental notions (ctd.)

So one also can say: a strategy profile is weakly Pareto efficient if there is no other strategy profile in which each player is better off. And a strategy profile is strongly Pareto efficient if there is no other strategy profile in which at least one player is better off and no player is worse off.

Thus there are two Pareto efficiency notions. Both, of course, are interesting. May be the most important notion in economics is that of Pareto-efficiency. If one speaks about Pareto efficiency one usually means strong Pareto-efficiency. Note that weak Pareto efficiency is a simpler notion than strong Pareto efficiency.

Fundamental notions (ctd.)

A strategy profile is

- fully cooperative if the total payoff in this strategy profile is maximal.

A Prisoners' dilemma game is a game with a strictly dominant equilibrium that is Pareto inefficient.

Finally: a zero-sum game is a game where the total payoff is zero in each strategy profile. And an antagonistic game is a zero-sum game with two players.

Many parlour games are antagonistic games.

Solution concepts

The aim of game theory is to understand/predict how games will be played. Here so-called solution concepts play a role. For bimatrix games and more general for games in strategic form (Lesson 3), the above notions of strictly dominant equilibrium and Nash equilibrium are such concepts. We shall later see other ones.

Examples

1. Determine the dominant strategies, the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 2;4 & 1;4 & 4;3 & 3;0 \\ 1;1 & 1;2 & 5;2 & 6;1 \\ 1;2 & 0;5 & 3;4 & 7;3 \\ 0;6 & 0;4 & 3;4 & 1;5 \end{pmatrix}.$$

No dominant strategies. No strictly dominant strategies. Nash equilibria: strategy profiles $(1,1),(1,2),(2,2)$ and $(2,3)$.

Attention: a notation as $(2,3)$ here above denotes the strategy profile where player 1 plays row 2 and player 2 plays column 3. So it deals with strategies and not with payoffs (which in strategy profile $(2,3)$ are 5 for player 1 and 2 for player 2).

Examples (ctd.)

2. Determine the dominant strategies, the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 6;1 & 7;1 & 6;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}$$

Strictly dominant strategies for player 1: strategy 1. Dominant strategy for player 1: strategy 1. Strictly dominant strategies for player 2: none. Dominant strategy for player 2: none. Nash equilibria: strategy profile $(1, 3)$.

Examples (ctd.)

3. Determine the dominant strategies, the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 3;-1 & 3;1 & 6;0 \\ 1;0 & 3;1 & 8;0 \\ 5;2 & 4;1 & 8;2 \end{pmatrix}.$$

No strictly dominant strategies. Dominant strategy for player 1: strategy 3. Dominant strategy for player 2: none. Nash equilibria: strategy profiles $(3,1)$ and $(3,3)$.

Examples (ctd.)

4. Determine the dominant strategies, the strictly dominant strategies and the Nash equilibria for

$$\begin{pmatrix} 1;0 & 1;2 & 0;4 \end{pmatrix}.$$

Strictly dominant strategies for player 1: strategy 1. Dominant strategies for player 1: strategy 1. Strictly dominant strategies for player 2: strategy 3. Dominant strategy for player 2: strategy 3. Nash equilibria: strategy profile $(1, 3)$.

Examples (ctd.)

5. Determine the strongly and weakly Pareto efficient strategy profiles for

$$\begin{pmatrix} 1;0 & 3;1 & 6;0 \\ 2;1 & 4;1 & 8;1 \end{pmatrix}.$$

   Weakly Pareto efficient strategy profiles: (1,2), (2,1), (2,2), (2,3). Strongly Pareto efficient strategy profiles: (2,3).

Examples (ctd.)

6. Determine the strongly and weakly Pareto efficient strategy profiles for
$$\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}.$$

   Weakly Pareto efficient strategy profiles:
   $(1,1),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$.
   Strongly Pareto efficient strategy profiles:
   $(1,1),(1,3),(2,1),(3,2),(3,3)$.

   For example: $(2,2)$ is not strongly Pareto efficient as $(3,3)$ is a Pareto improvement of $(2,2)$.

Examples (ctd.)

7. Determine the fully cooperative strategy profiles for

$$\begin{pmatrix} 1;0 & 1;-4 & 0;1 \\ 1;1 & 0;2 & -2;0 \end{pmatrix}.$$

Fully cooperative strategy profiles: $(2,1), (2,2)$.

Examples (ctd.)

8. Determine the strictly dominant equilibria for the following game. Is the game a prisoner's dilemma game?

$$\begin{pmatrix} 1;0 & -1;4 & 0;2 \\ 0;6 & 0;2 & 0;3 \end{pmatrix}.$$

No player has a strictly dominant strategy; therefore there is no strictly dominant equilibrium and the game is not a prisoners' dilemma.

Examples (ctd.)

9. Determine the strictly dominant equilibria for the following game. Is the game a prisoner's dilemma game?

$$\begin{pmatrix} -1;-1 & 2;0 \\ 0;2 & 3;3 \end{pmatrix}.$$

Both players have a strictly dominant strategy: their second one. So $(2,2)$ is a strictly dominant equilibrium. As $(2,2)$ is weakly Pareto efficient (and even strongly Pareto efficient), the game is not a prisoners' dilemma game.

Examples (ctd.)

10. Determine the dom. and strictly dom. strategies, the strictly dom. equilibria, the Nash eq. the weakly and strongly Pareto eff. strat. profiles and the fully coop. strat. prof. for

$$\begin{pmatrix} -1;0 & -1;1 & 0;0 \\ 2;-2 & -3;3 & -1;3 \\ 4;-3 & 5;-5 & 1;-7 \\ 3;-3 & 3;-5 & -6;8 \end{pmatrix}.$$

Strictly dom. strategies for player 1: strategy 3. Dom. strategies for player 1: strategy 3. Strictly dom. strategies for player 1: none. Dom. strategies for player 1: none. Strictly dom. equilibria: none. Nash eq. strat. profile $(3,1)$. Weakly Pareto eff. strat. prof.: $(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(4,1),(4,3)$.

Examples (ctd.)

Strongly Pareto efficient strat. prof.: $(1,3),(2,1),(2,3),(3,1),(3,2),(4,3)$. Fully cooperative strat. prof.: $(2,3),(4,3)$.

For example: $(4,2)$ is not weakly Pareto efficient as $(3,1)$ is an unanimous Pareto improvement of $(4,2)$. And $(1,2)$ is not strongly Pareto efficient as $(2,3)$ is a Pareto improvement of $(1,2)$.

11. Determine the weakly and strongly Pareto efficient strategy profiles for

$$\begin{pmatrix} 3;8 & 4;8 & 2;3 \\ 1;7 & 2;6 & 8;1 \\ 3;4 & 4;4 & 2;2 \\ 1;1 & 1;-1 & 1;-1 \end{pmatrix}.$$

Weakly: (1,1), (1,2), (2,3), (3,2). Strongly: (1,2) (2,3).

Some simple relations

Here are some simple relations between the fundamental notions.

- Each strictly dominant strategy is a dominant strategy.
- A player can have at most one strictly dominant strategy, implying that a game can have at most one strictly dominant equilibrium.
- A strongly Pareto efficient strategy profile is weakly Pareto efficient, implying that a weakly Pareto inefficient strategy profile is strongly Pareto inefficient.
- A fully cooperative strategy profile is strongly Pareto efficient.

Some simple relations (ctd.)

It should be clear that the relations in the the first three bullets hold. In Exercises 2 You have to prove that the relation in the last bullet holds. Please, check in the above examples that these relations indeed hold true.

Concrete games revisited

Concerning the five concrete games that we dealt with in Lesson 1 (i.e. Tic-tac-toe, Hex, Cournot Oligopoly, Hotelling Game, Nim) only the Hotelling Game is a bimatrix game. The Cournot OLigopoly is not a bimatrix game as each player there has infinite many strategies.
We shall now show how the Hotelling Game can be represented as a bimatrix game. In the next lesson the notion of game in strategic form is introduced, which allows us to handle the Cournot Oligopoly. We have to wait until Lesson 4 for the other three remaining games.

Hotelling Game revisited (ctd)

Please, if needed, review Lesson 1 for the definition of the Hotelling Game. Now consider this game of three sites: $0, 1$ and 2. (So $m = 2$)
This game can be represented as a 3×3-bi-matrix game with, for player 1 at the first row strategy 0, at the second row strategy 1, at the third row strategy 2. And with the same convention for player 2.

If You do this correctly (and do it!), then You find

$$\begin{pmatrix} 3/2; 3/2 & 1; 2 & 3/2; 3/2 \\ 2; 1 & 3/2; 3/2 & 2; 1 \\ 3/2; 3/2 & 1; 2 & 3/2; 3/2 \end{pmatrix}.$$

Hotelling Game revisited (ctd.)

The game has a unique Nash equilibrium: the strategy profile $(1, 1)$. So the players (may be icecream sellers) locate in the middle.

In Exercises 2 You will show that this result does not depend on the the number of sites.