

Advanced Microeconomics

P. v. Mouche

Exercises 3

Exercise 1 Given the following bimatrix game:

$$\begin{pmatrix} 3; 8 & 4; 8 & 2; 3 \\ 1; 7 & 2; 6 & 8; 1 \\ 3; 4 & 4; 4 & 2; 2 \\ 1; 1 & 1; -1 & 1; -1 \end{pmatrix}.$$

- Determine the best reply correspondences.
- Determine the dominant strategies and the strictly dominant strategies.
- Determine the strategy profiles that survive the procedure of elimination of strongly dominated strategies.
- Determine the Nash equilibria.
- Determine the strongly Pareto-efficient strategy profiles and the weakly Pareto-efficient strategy profiles.

Exercise 2 Consider the bimatrix game

$$\begin{pmatrix} 5; 12 & 0; 0 \\ 0; 0 & 10; 4 \end{pmatrix}.$$

- Determine, in case of pure strategies, the best reply correspondences, and if it/they exist, the strictly dominant Nash equilibrium, the iterative not strongly dominated equilibrium and the Nash equilibria.
- Determine, in case of mixed strategies, the best reply correspondences and the Nash equilibria.

Exercise 3 Consider a game in strategic form with two players. The game is called ‘strictly competitive’ if for all strategy profiles (x_1, x_2) and (y_1, y_2)

$$f_1(x_1, x_2) \geq f_1(y_1, y_2) \Leftrightarrow f_2(x_1, x_2) \leq f_2(y_1, y_2).$$

The game is called a ‘constant-sum game’ if there exists a number $c \in \mathbb{R}$ such that for each strategy profile (x_1, x_2) it holds that $f_1(x_1, x_2) + f_2(x_1, x_2) = c$.

- Show that each constant-sum game is strictly competitive.
- Show that in a strictly competitive game each strategy profile is strongly Pareto efficient.

Exercise 4 Make Exercise 7.4 (on strong domination) from the text book.

Exercise 5 Consider a homogeneous cournot-oligopoly, i.e. a game in strategic form where each player i has as strategy set $X_i = [0, m_i]$ with $m_i > 0$ and a payoff function f_i of the form

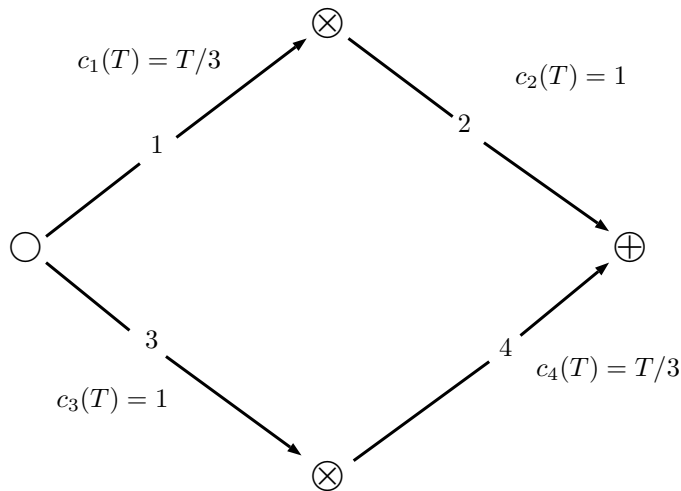
$$f_i(x_1, \dots, x_n) = p(x_1 + \dots + x_n)x_i - c_i(x_i),$$

where $p : [0, m_1 + \dots + m_n] \rightarrow \mathbb{R}$ and $c_i : X_i \rightarrow \mathbb{R}$. Here x_i is called ‘production level’ of firm i . f_i is called ‘profit function’ of firm i , p ‘inverse demand function’ and c_i ‘cost function’ of firm i . A Nash equilibrium of this game also is called cournot equilibrium.

Suppose that p is decreasing, concave and twice differentiable and that each c_i is convex and twice differentiable.

- Prove that each conditional profit function is concave.
- Prove, using the Nikaido-Isoda theorem, that the game has at least one cournot equilibrium.

Exercise 6 Consider the following traffic network.



- Identify for each commuter the strategies.
- Further suppose that there are 2 commuters. Represent this game as a bimatrix game.
- Determine the Nash equilibria.

Short solutions.

Solution 1 a. $R_1(1) = \{1, 3\}$, $R_1(2) = \{1, 3\}$, $R_1(3) = \{2\}$, $R_2(1) = \{1, 2\}$, $R_2(2) = \{1\}$, $R_2(3) = \{1, 2\}$, $R_2(4) = \{1\}$.

b. Dominant strategies for player 1: do not exist.

Dominant strategies for player 2: the first.

Strictly dominant strategies: do not exist.

c. Step 1:

$$\begin{pmatrix} 3; 8 & 4; 8 \\ 1; 7 & 2; 6 \\ 3; 4 & 4; 4 \end{pmatrix}.$$

Step 2:

$$\begin{pmatrix} 3; 8 & 4; 8 \\ 3; 4 & 4; 4 \end{pmatrix}.$$

d. They are (1, 1) (i.e. row 1 and column 1), (1, 2), (3, 1), (3, 2).

e. Strongly: (1, 2) (2, 3). Weakly: (1, 1), (1, 2) (2, 3), (3, 2).

Solution 2 a. $R_1(1) = \{1\}$, $R_1(2) = \{2\}$, $R_2(1) = \{1\}$, $R_2(2) = \{2\}$.

No strictly dominant equilibrium and no iterative not strongly dominated equilibrium.

Nash equilibria: (1, 1) and (2, 2).

b. Expected payoff functions: $\bar{f}_1(p; q) = (15q - 10)p + 10 - 10q$ and $\bar{f}_2(p; q) = (16p - 4)q + 4 - 4p$.

Best reply correspondences: $\bar{R}_1(q) = \begin{cases} \{1\} & \text{if } q > 2/3, \\ [0, 1] & \text{if } q = 2/3, \\ \{0\} & \text{if } q < 2/3 \end{cases}$ and $\bar{R}_2(p) = \begin{cases} \{1\} & \text{if } p > 1/4, \\ [0, 1] & \text{if } p = 1/4, \\ \{0\} & \text{if } p < 1/4. \end{cases}$

Nash equilibria: $p = 1/4, q = 2/3$; $p = 0, q = 0$; $p = 1, q = 1$.

Solution 3 a. For a constant-sum game we have $f_1 + f_2 = c$. So $f_1(a, b) \geq f_1(c, d) \Leftrightarrow c - f_2(a, b) \geq c - f_2(c, d) \Leftrightarrow f_2(a, b) \leq f_2(c, d)$. Thus the game is strictly competitive.

b. By contradiction suppose \mathbf{a} is strongly Pareto-inefficient. Then there exists a Pareto improvement \mathbf{b} of \mathbf{a} . So then there exists a player, say 1, who at \mathbf{b} has a greater payoff, and the other player does not have a smaller payoff. Thus $f_1(\mathbf{b}) > f_1(\mathbf{a})$ and $f_2(\mathbf{b}) \geq f_2(\mathbf{a})$. As the game is strictly competitive, we have $f_1(\mathbf{b}) \leq f_1(\mathbf{a})$, a contradiction.

Solution 4 We consider the bimatrix-game $(A; B)$ with $B = \begin{pmatrix} 0 & -3 & -4 \\ 4 & 5 & 8 \end{pmatrix}$. Then the second pure strategy of player 2 is not strongly dominated by a pure strategy. But for each mixed strategy (p_1, p_2) of player 1 we have $f_2((p_1, p_2), (1/2, 0, 1/2)) = 6 - 8p_1 > 5 - 8p_1 = f_2((p_1, p_2), (0, 1, 0))$. Therefore the second pure strategy of player 2 is strongly dominated by his mixed strategy $(1/2, 0, 1/2)$.

Solution 5 a. Consider a player i . Fix a strategy of each other players. Let a be the sum of these strategies. For this situation the conditional payoff function is

$$g(x_i) = p(x_i + a)x_i - c_i(x_i).$$

Therefore

$$g''(x_i) = p''(x_i + a)x_i + 2p'(x_i + a) - c_i''(x_i).$$

As $p'' \leq 0, p' \leq 0$ and $c_i'' \leq 0$, it follows that $g'' \leq 0$. Thus g is concave.

b. Having part a, apply the Nikaido-Isoda equilibrium existence result (from the slides).

Solution 6 a. Strategy 1 is route choice $\{1, 2\}$. Strategy 2 is route choice $\{3, 4\}$

b. $\begin{pmatrix} 5/3; 5/3 & 4/3; 4/3 \\ 4/3; 4/3 & 5/3; 5/3 \end{pmatrix}$.

c. This game has two Nash equilibria: (1, 2) and (2, 1). In each Nash equilibrium each player has costs $4/3$.