Game Theory:
Cooperative Games

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For quality of life

O. Morgenstern, J. v. Neumann, L. Shapley, J. Nash

Cooperative Games

Plan for today

Cooperative vs. non-cooperative games

Axiomatic method

Two cooperative solution concepts

- Core
- Shapley Value

Friday

Bargaining solutions

Cooperative Games

		Player 2	
		left	right
Player 1	up	2, 2	-1, 3
	down	3, -1	0, 0

- Individual rationality
- Collective rationality

In a cooperative game players can make binding agreements

Consequently, (Pareto) efficient outcomes can be achieved.

Cooperative Games

Non-cooperative	Cooperative
Individual rationality	Individual and collective rationality
Sequence of choice can matter, game trees	Sequence of choice is irrelevant, no tree structure
Physical strategies matter	Strategies are implicit or absent
Solution found by assessing strategic choices	Solution found by assessing characteristics of payoffs, i.e axiomatic method
Solution concept: Nash equilibrium and refinements	Many different solution concepts
Frequently inefficient solutions	Efficiency usually guaranteed
Usually meant to describe or predict outcomes	Usually meant as a normative approach

Axiomatic approach

Axioms are postulates taken to be true
(also called “first principles” or “premises”)

They form the bases of deductive systems

Major developments around 1900 with the rise of mathematical logic (... but it goes back Euclid).

Example: Peano’s axioms of number theory

- 0 is a natural number.
- For every natural number x, $x = x$.
- For all natural numbers x, y, if $x = y$, then $y = x$.
- ...

Axiomatic approach in cooperative game theory

Example: Axiomatic bargaining or cost sharing

e.g.

- Anonymity
- Symmetry
- Monotonicity

Solutions are characterised by their properties.

Preliminaries: Notation in set theory

Set of players	$N = \{1, 2, \ldots, i, \ldots, n\}$
Subset	$S \subseteq N$
Union	$S \cup T$
Intersection	$S \cap T$
Empty set	$\varnothing$
S without T	$S \setminus T$
Complement	$N \setminus S$
Power set	$\mathcal{P}(N)$

Preliminaries

TU games (transferable utility games), utility is linear in money.

Coalitional games

We have a set of players N.

Subsets of players are called $S \subseteq N$.

Payoffs are defined for coalitions.

We call $v(S)$ the worth of the coalition.

Individual payoffs x_i must satisfy $\sum_{i \in S} x_i \leq v(S)$.

A game is a pair (N, v)

(market games, cost sharing games, voting games)

Core of TU games

(1) $v(\emptyset) = 0$

(2) $v(S \cup T) \geq v(S) + v(T)$; for $S \cap T = \emptyset$

(1) is a normalisation.

(2) is the superadditivity condition

$v(S)$ is the payoff that a coalition can insure for itself; the maximin value.

(3) $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$

Condition (3) defines a convex game, but also

(4) $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for $S \subset T$.

Core of TU games

An imputation is a payoff vector *x* that distributes the grand coalition payoff between players satisfying individual rationality and Pareto optimality.

$$\sum_{i \in N} x_i = v(N); \text{ and for all } i,\ x_i \geq v(\{i\}).$$

Domination: *x* dominates *x'* in *S* if for all $i \in S,\ x_i \geq x_i'$ and the inequality is strict for some $i \in S$.

Core of TU games

The core is the set of all undominated imputations.

For a imputation (payoff vector) in core it must hold that there is no $S \subset N$ such that

$$v(S) > \sum_{i \in S} x_i$$

Thus, an imputation in the core is individually and collectively rational.

The core of game Γ is a set

$$C(\Gamma) = \{x : v(S) - \sum_{i \in S} x_i \leq 0, \text{ for all } S \subset N\}$$

Core of TU games

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Any solution in the core cannot be blocked by any coalition.

Shapley Value

For a coalitional game with characteristic function $v(S)$, the Shapley value assigns to each player $i \in N$

$$\varphi_i(v) = \sum_{S \subset N \backslash \{i\}} \frac{(s-1)!(n-s)!}{n!} \left(v(S \cup \{i\}) - v(S) \right)$$

where s and n are the numbers of the members of S and N, respectively. We have

$$\sum_{i \in S} \varphi_i(v) = 1.$$

Shapley Value

The Shapley value is the unique imputation that satisfies

- Group rationality
- Symmetry (the order of players does not matter)
- Additivity $\varphi_i(v+w) = \varphi_i(v) + \varphi_i(w)$

It also satisfies the Null player condition

$\varphi_i(v) = 0$ if for all S $v(S \cup \{i\}) - v(s) = 0.$

Bargaining

Nash Bargaining: Axioms

Bargaining problem: A set of possible outcomes and a threatpoint (S,d)

A solution should satisfy:

A1: Independence of utility transformations

A2: Symmetry

A3: Independence of irrelevant alternatives

A4: Pareto optimality

Nash, J. (1950) The Bargaining Problem. Econometrica 18, 286-295.

Roth, A. (1979) Axiomatic Models of Bargaining. Lecture Notes in Economics and Mathematical Systems 170. Berlin: Springer.

Bargaining XXXX

Kalai – Smorodinski solution

Bargaining problem: A set of possible outcomes and a threatpoint (S,d)

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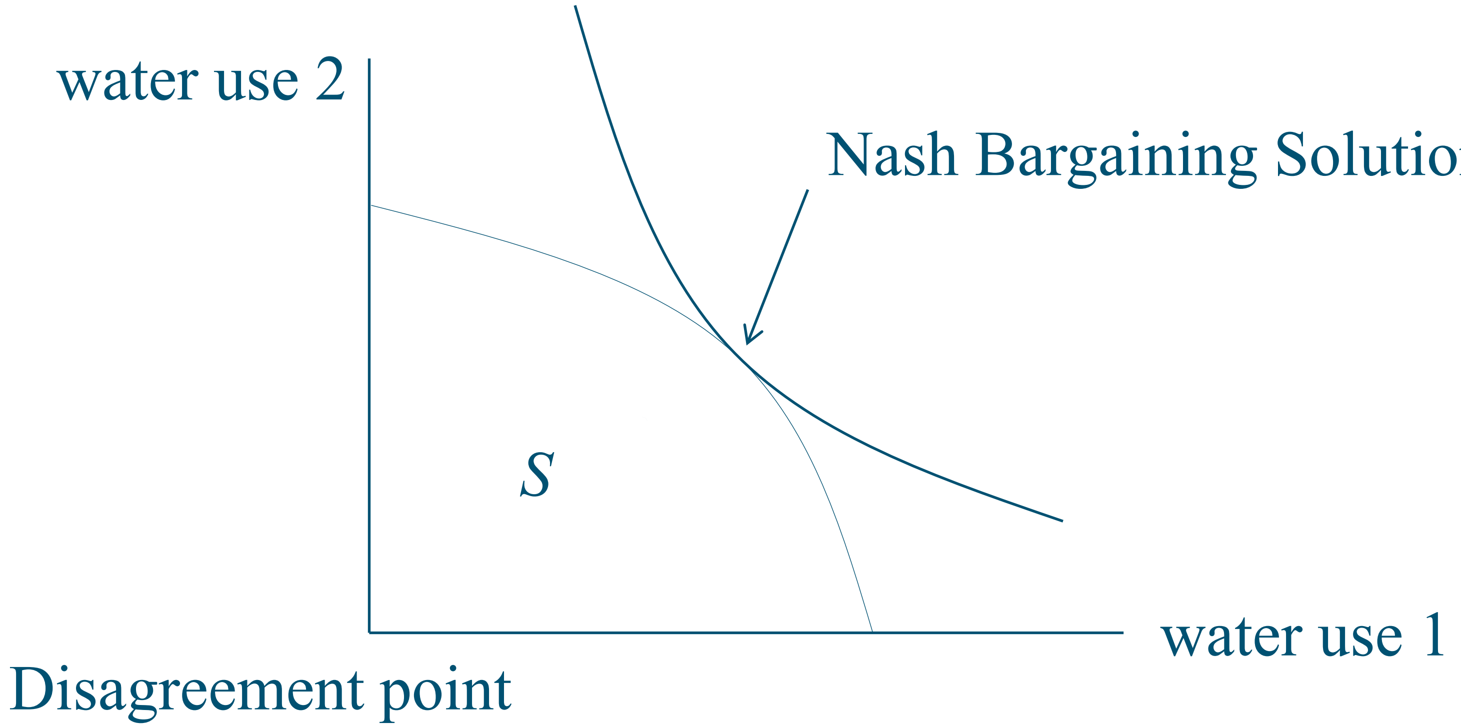
Nash Bargaining Solution



Nash Bargaining solution

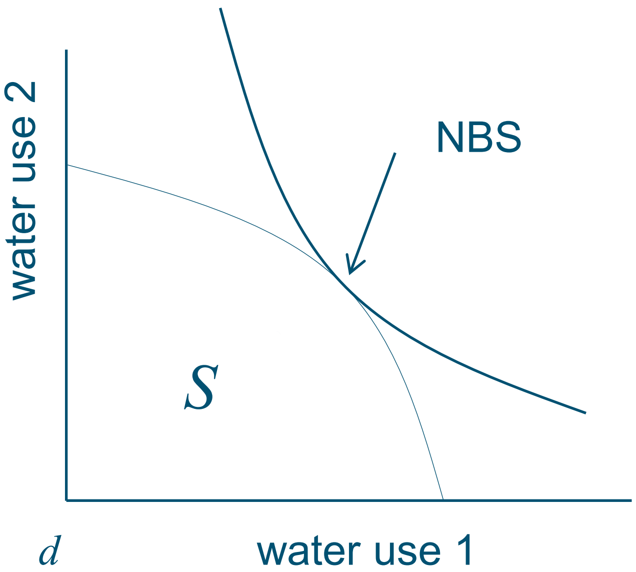
Nash Bargaining Solution

$$\max \underbrace{(u_1 - d_1)^{\alpha_1} \cdot (u_2 - d_2)^{\alpha_2}}_{\text{Nash product}}$$

α_1, α_2 are the bargaining weigths

$$\sum_{i \in N} \alpha_i = 1$$

The solution satisfies (i) Invariance to Equivalent Utility Representations, (ii) Symmetry, (iii) Independence of Irrelevant Alternatives, and (iv) Pareto efficiency.



The Nash programme

Rubinstein's bargaining game:

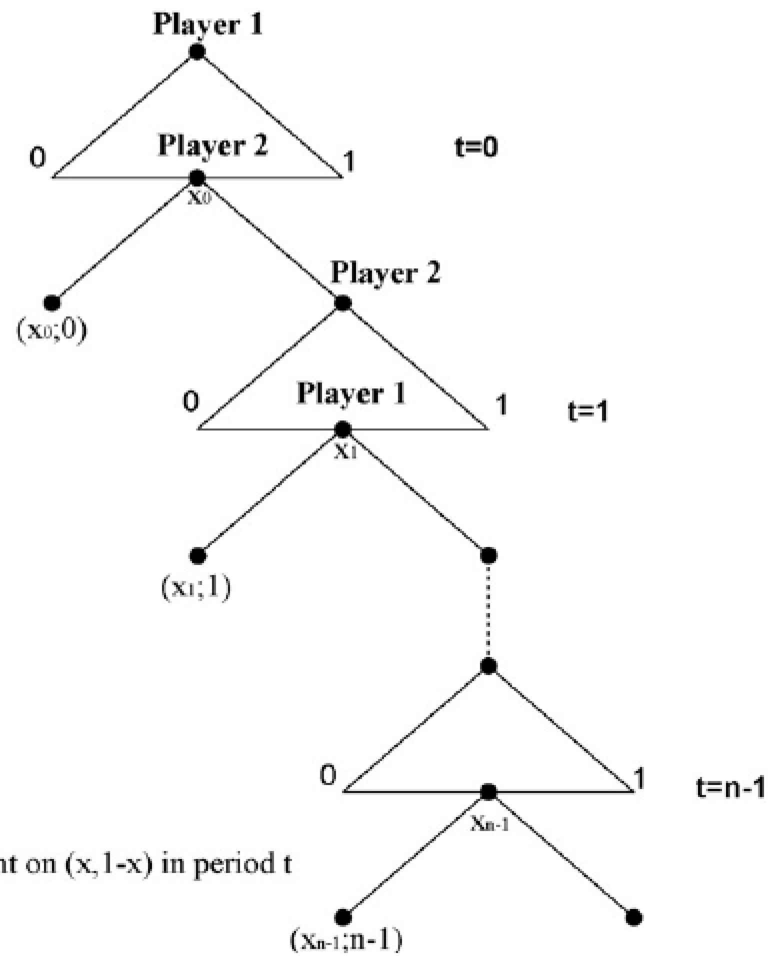
The alternating offer model:

Players' shares:

$$\pi_1 = \frac{1-\delta_2}{1-\delta_1\delta_2};$$

$$\pi_2 = 1 - \frac{1-\delta_2}{1-\delta_1\delta_2} = \frac{1-\delta_1}{1-\delta_1\delta_2}$$

Implements the NBS



(x;t) : agreement on (x,1-x) in period t

Rubinstein, A. (1982) Perfect Equilibrium
in a Bargaining Model. Econometrica 50, 97-109.