Problem 1 *Consider a cost sharing game* (N, c) *with* $N = \{1, 2, 3\}$ *and the cost (charateristic) function c given by*

$c(\{1\}) = 120; c(\{2\}) = 140; c(\{3\}) = 120; c(\{1,2\}) = 170; c(\{1,3\}) = 160; c(\{2,3\}) = 190; c(\{1,2,3\}) = 265.$

a) Describe in your own words what function c tells us. b) Show that the core of (N, c) *is empty.*

Problem 2 *Consider a cooperative 2-player bargaining problem* (S, d). *Denote a solution to the bargaining problem by* $f((S, d)) = (x_1, x_2)$. *Let S be given by the triangle [(0,0),(0,4),(8,0)] in the players' payoff space and let* $d = (1, 2)$.

a) Write down the Nash product.

b) Maximizing the Nash product, subject to $(x_1, x_2) \in S$ *gives* $\frac{9}{8}$. *Use this and the given information to determine the bargaining solution.*

c) Argue why the gains from bargaining are unequally divided.

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Problem 3 *One interpretation of an egalitarian solution in a cost sharing game with subadditive costs is equal sharing of surplus. Consider a cost sharing game* (N, c) *with* $N = \{1, 2\}$ *and the cost (charateristic) function c. The surplus is defined as* $W = c(\{1, 2\}) - c(\{1\}) - c(\{2\})$. *Then egalitarian surplus sharing means* $(x_1^e, x_2^e) = (c(1) - \frac{W}{2}, c(2) - \frac{W}{2})$.

a) Calculate the egalitarian surplus sharing solution when $c(\{1\}) = 120; c(\{2\}) = 140; c(\{1,2\}) = 170$.

b) Is this solution in the core? Why?

c) How can egalitarian surplus sharing be generalized to three players? And to n players?

d) Consider now $c(\{1\}) = 120; c(\{2\}) = 140; c(\{1,2\}) = 170$, *as before and a third player such that* $c(\{3\}) = 120; c(\{1,3\}) = 160; c(\{2,3\}) = 190; c(\{1,2,3\}) = 255$. *Calculate the payoffs for egalitarian surplus sharing.*

e) Show that the solution for d) is not in the core.

f) Find the core of the cost sharing game.

Answers

a) The surplus is $260 - 170 = 90$. This is shared equally by both players. Thus, both players costs are reduced by 45 to 75 and 95, respectively.

b) This solution is in the core as no player can do better by breaking away from the coalition.

c)The surplus for n players would be defined as $\sum_{i \in N} c_i - c(N)$. This is to be distributed equally among players

d) The surplus is $380 - 255 = 125$. Each player's costs are reduced by $\frac{125}{3}$.

e) The cost allocation under egalitarian surplus sharing is $c(\{1\}) = 78\frac{1}{3}; c(\{2\}) = 98\frac{1}{3}; c(\{3\}) = 78\frac{1}{3}$. Notice that this solution is blocked by coalition $\{1, 2\}$ whose cost is 170 while that would have to pay $176\frac{2}{3}$ in the suggested solution.

f) All (x_1, x_2, x_3) that satisfy $(x_1 + x_2 + x_3) = 255; c(\{1\}) \leq 120; c(\{2\}) \leq 140; c(\{3\}) \leq 120; c(\{1, 2\}) \leq 170; c(\{1, 3\}) \leq 160; c(\{2, 3\}) \leq 190$ are in the core.

Problem 4 *Consider a two player cost sharing problem with characteristic function* $c(\{1\}) = -120; c(\{2\}) = -140; c(\{1, 2\}) = -170$..

a) Describe the situation as a bargaining game, i.e. determine the disagreement point and the bargaining set.

b) Find the Nash bargaining solution.

c) Argue that it must be in the core.

Answers

a) We take the costs to be paid as singleton players as the payoffs of the disagreement point. Hence $d = (-120, -140)$. The bargaining set will distribute the cost under cooperation. Hence $S = \{(x_1, x_2) | x_1 + x_2 \leq -170, x_1 \geq -120, x_2 \geq -140\}$.

b) When we maximize the Nash product $(x_1 - d_1)(x_2 - d_2)$ under the constraint that $x_1 + x_2 \leq -170$ we obtain $c(\{1\}) = -75; c(\{2\}) = -95$ as the solution.

c) We can argue that this solution cannot be blocked by any of the two players as they would have higher cost (lower payoff) when acting on their own.