Advanced Microeconomics: Game Theory

Lesson 4: Games in Extensive Form

P. v. Mouche

2020, Period 1

What You will learn

After studying Lesson 4, You

- should know what we mean by a game in extensive form.
- should be able to transform a game in extensive form into a game in strategic form by means of normalisation.
- should be able to predict how various antagonistic games with a value will be played.

Appetizer

Consider the following parlour games: tic-tac-toe, chess, 8×8 checkers, hex and nim. They have in common that they are antagonistic games, i.e. zero-sum games with two players where player 1 makes the first move. (Also remember that a concrete hex game depends on the board size.)

We already know that tic-tac-toe, chess and 8×8 checkers ends with a winner (and a loser) or with a draw. We also know that hex cannot end in a draw and so each hex game ends with a winner. (The nim game will be explained and dealt with in the assignment for part 2.)

Appetizer (ctd.)

With the theory in this lesson we shall see that we can make very strong predictions how rational intelligent players will play these games. The reason is that all these games game have a so-called value which will be realized if each player plays a so-called optimal strategy.

Of course, these notions have to be defined, and we shall do so. For the moment take this intuitive definition:

- Value : the outcome of the game in the case of two rational intelligent players.
- Optimal strategy for a player: a strategy that guarantees this player at least the value.

Appetizer (ctd.)

Here are the results for these games:

	tic-t.-t.	chess	checkers	hex	nim
value	draw	not known	draw	1 wins	known
opt. strat.	known	not known	known	not known	known

It is interesting to note that we claimed that all these games have a value, but that the actual value up to now for chess is not known. Also interesting to note is that we know that player 1 can win each hex game, but that we do not know how he could do this.

Game in extensive form

The setting in the remainder of this lesson always is a non-cooperative one with complete information and perfect information and no chance moves.

A formal definition of a game in extensive form is quite technical: see Definition 7.13 in the text book. I shall proceed here now less formally.

Game in extensive form (ctd.)

A game in extensive form can be represented by a game tree.
In such a tree

- there are nodes (also called histories): end nodes, decision nodes and a unique initial node;
- there are a (directed) branches;
- there are payoffs at the end nodes;
- each non-initial node has exactly one predecessor;
- no path in the tree connects a node with itself.

We further always assume that the game is finite, i.e. that the number of nodes and branches is finite.

Notion of strategy

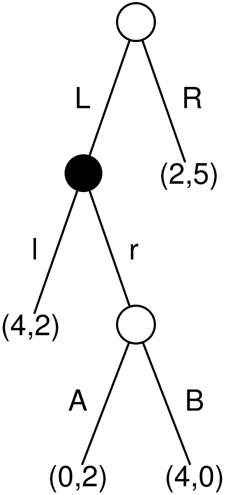
Given a game in extensive form, for a player the notion of completely elaborated plan of playing is defined. One may consider it as a piece of paper on which that player writes which move he will make if it is his turn.

A related notion is that of strategy of a player: this is specification at each decision node how to move.

Note that a strategy may be much more than a completely elaborated plan of play: the player also has to specify his moves at nodes which never may be reached. This sounds strange. In Lesson 5 we shall see why this notion of strategy is very important in order to understand how games will be played.

Example

Here is an example of a game tree:



Example (ctd.)

In this game tree, the nodes where player 1 moves are presented by the $\circ$ symbol and those for player 2 by the $\bullet$ symbols.

The possible moves are denoted by the symbols L, R, l, r, A, B.

The payoffs are at given at the end nodes (for which no symbol is used).

Example (ctd.)

In this game

- Player 1 has 4 strategies: *LA*, *LB*, *RA* and *RB*.
- Player 2 has 2 strategies: *l* and *r*.
- Playing *R* is a completely elaborated plan of play for player 1, but it is not a strategy. If You do not understand this, then look back to the definition of strategy!

This game will be dealt with further in Exercise 3 belonging and in Lesson 5.

Normalisation

Out of a given a game in extensive form, one can make in a natural way a game in strategic form: just consider for each player all possible strategies and calculate the payoffs at each possible strategy profile.

This transformation of a game in extensive form into a game in strategic form is called normalisation .

As a game in strategic form is a game with imperfect information, normalisation destroys the perfect information.

Normalisation makes that all terminology and results for games in strategic form now also applies to games in extensive forms.

Solving from the end to the beginning

As an introduction to the next lesson, we now consider the following game between two players. There is a pillow with 21 matches. They alternately remove $1, 3$ or 4 matches from it. (Player 1 begins.) The player who makes the last move wins. How this game will be played?

The idea to handle this question is to analyse the game ‘from the end to the beginning’:

Solving from the end to the beginning (ctd.)

If there are 0 matches left, then the player who has to play loses. So 0 is a losing position

If there is 1 match left, then the player who has to play wins. So 1 is a winning position.

If there are 2 matches left, then the player who has to play has to remove one match and a position which 1 match remains. As this is a winning position, 2 is a losing position.

If there are 3 matches left, then the player who has to play can remove these matches and then wins. So 3 is a winning position.

Solving from the end to the beginning (ctd.)

If there are 4 matches left, then the player who has to play can remove these matches and then wins. So 4 is a winning position.

If there are 5 matches left, then the player who has to play can remove 3 matches and a position which 2 matches remains. As this is a losing position, 5 is a winning position.

If there are 6 matches left, then the player who has to play can remove 4 matches and a position which 2 matches remains. As this is a losing position, 6 is a winning position.

If there are 7 matches left, then the player who has to play always will end up in a winning position. So 7 is a losing position.

Solving from the end to the beginning (ctd.)

And so on: the loosing positions are $0, 2, 7, 9, 14, 16, 21, \ldots$, i.e. the numbers that have remainder 0 or 2 when divided by 7.

Because $100/7$ has remainder 2, 100 is a loosing position. Conclusion: player 2 always can win the game, implying that he has a winning strategy.

Hex-game revisited

Finally, in this lesson, we show here that in the Hex-game player 2 cannot have a winning completely elaborated plan of playing. This we do by showing that if player 2 would have such a plan, player 1 also would have one, which is absurd.

The clever proof that is presented here is a so-called ‘strategy stealing argument’.

Hex-game revisited (ctd.)

So suppose that player 2 has a winning completely elaborated plan of playing, which we will call S. Now consider the following completely elaborated plan of playing for player 1.

Player 1 makes his first move at random. Thereafter he should pretend to be player 2, 'stealing' the second player's elaborated plan of play S, and follow strategy S, which by hypothesis will result in a victory for him. What, of course, we still have to show is that this is a legitimate completely elaborated plan of playing. Well, if strategy S calls for him to move in the hexagon that he chose at random for his first move, he should choose at random again. This will not interfere with the execution of S.

Finally, note that the above strategy of player 1 is at least as good as strategy S of player 2 since having an extra marked square on the board is never a disadvantage in hex.