Advanced Microeconomics: Game Theory

Lesson 3: Games in Strategic Form

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2020, Period 1

What You will learn

After studying Lesson 3, You

- should understand, for games in strategic form, the introduced game theoretic vocabulary formed by the fundamental notions.
- should know how to make predictions by using solution concepts.
- should able to deal with mixed strategies for bimatrix games.

Main mathematical types

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- Games in strategic form.
- Games in extensive form. (We deal with in next lessons.)
- Games in characteristic function form. (We will not deal with as they belong to cooperative game theory.)

Game in strategic form

Definition

Game in strategic form, specified by

- n players: $1, \ldots, n$.
- for each player i a strategy set (or action set) X_i. Let $\mathbf{X} := X_1 \times \cdots \times X_n$: set of strategy profiles.
- for each player i payoff function $f_i : \mathbf{X} \to \mathbb{R}$.

Interpretation: players choose simultaneously and independently a strategy.

Game in strategic form (ctd.)

A game in strategic form is called finite if each strategy set X_i is finite.

Of course, in the case of two players a finite game in strategic form can be represented as a bimatrix game.

Besides the choices in red on the slides in Lesson 1 concerning real-world-types, we further assume complete information, static game and for the moment no chance moves.

Please note that a game in strategic form is a game with imperfect information as the moves are simultaneously.

Some concrete games.

$$\begin{pmatrix} 0;0 & -1;1 & 1;-1 \\ 1;-1 & 0;0 & -1;1 \\ -1;1 & 1;-1 & 0;0 \end{pmatrix}$$

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Stone-paper-scissors

Some concrete games (ctd).

Cournot oligopoly : n arbitrary, $X_i = [0, m_i]$

$$f_i(x_1, \ldots, x_n) = p(x_1 + \cdots + x_n)x_i - c_i(x_i).$$

p: price function, c_i: cost function.

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Transboundary pollution game : n arbitrary, $X_i = [0, m_i]$

$$f_i(x_1, \ldots, x_n) = \mathcal{P}_i(x_i) - \mathcal{D}_i(T_{i1}x_1 + \cdots + T_{in}x_n).$$

$\mathcal{P}_i$: production function, $\mathcal{D}_i$: damage cost function, T_{ij}: transport matrix coefficients.

Normalisation

Many games which are not defined as a game in strategic form can be represented in a natural way by normalisation as a game in strategic form.

For example, chess and tic-tac-toe. Indeed here $n = 2$, X_i is set of completely elaborated plans of playing of i and

$$f_i(x_1, x_2) \in \{-1, 0, 1\}.$$

We shall pick up 'normalisation' again in Lesson 4.

Fundamental notions

The fundamental notions for bimatrix games in Lesson 1 (dominant strategy, strictly dominant strategies, stritly dominant equilibrium, Nash equilibrium, strongly Pareto efficient strategy profile, weakly Pareto efficient strategy profile, fully cooperative strategy profile, prisoner's dilemma, zero-sum game) also make sense for an arbitrary game in strategic form. Their definition is exactly the same.

Please review these notions now!

Some additional fundamental notions will now be introduced.

Fundamental notions (ctd.)

- Conditional payoff function $f_i^{(\mathbf{z})}$ of player i: f_i as a function of x_i for fixed strategy profile $\mathbf{z}$ of the opponents.

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- Strongly (or strictly) dominated strategy of a player: a strategy of a player for which there exists another strategy that independently of the strategies of the other players always gives a higher payoff.

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- Strongly (or strictly) dominated strategy of a player: a strategy of a player for which there exists another strategy that independently of the strategies of the other players always gives a higher payoff.
- Weakly dominated strategy of a player: a strategy of a player for which there exists another strategy that independently of the strategies of the other players at least one time gives a higher payoff and never a smaller payoff.

Some simple relations

Here are some simple relations between the fundamental notions (already seen for bimatrix games in Lesson 1/Exercises 1).

- A player can have at most one strictly dominant strategy, implying that a game can have at most one strictly dominant equilibrium.
- A strongly Pareto efficient strategy profile is weakly Pareto efficient, implying that a weakly Pareto inefficient strategy profile is strongly Pareto inefficient.
- A fully cooperative strategy profile is strongly Pareto efficient.

Some simple relations (ctd.)

The definition of Nash equilibrium makes that in a Nash equilibrium each player plays a best reply against the strategies of the other players.

Formally: a strategy profile $\mathbf{e} = (e_1, \ldots, e_n)$ is a Nash equilibrium if and only if for each player i one has

$$e_i \in R_i(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n).$$

Solution concepts

The aim of game theory is to understand/predict how games will be played. Here so-called solution concepts play a role. For games in strategic form the following one are important.

- Strictly dominant equilibrium : strategy profile where each player has a strictly dominant strategy.

We already are familiar with these notions.

Solution concepts

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- Strictly dominant equilibrium : strategy profile where each player has a strictly dominant strategy.
- Nash equilibrium : strategy profile such that no player wants to change his strategy in that profile.

We already are familiar with these notions.

Here is a new one:

- Procedure of iterative (simultaneous) elimination of strongly dominated strategies
- Strategy profile that survives this procedure .
- If there is a unique strategy profile that survives the above procedure this strategy profile is called the iteratively not strongly dominated equilibrium .

Please see `https://www.youtube.com/watch?v=pC--lK8KNwo` for period 7:21-9:05 (and the text book).

Concerning this video: row 3 in the example therein deals with the elimination of weakly dominated strategies.

Example

Example

Determine the strictly dominant equilibria, the iteratively not strongly dominated equilibria and the Nash equilibria of the game

$$\begin{pmatrix} 2;4 & 1;4 & 4;3 & 3;0 \\ 1;1 & 1;2 & 5;2 & 6;1 \\ 1;2 & 0;5 & 3;4 & 7;3 \\ 0;6 & 0;4 & 3;4 & 1;5 \end{pmatrix}.$$

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Answer: no strictly dominant equilibrium. The procedure gives $\begin{pmatrix} 2;4 & 1;4 & 4;3 \\ 1;1 & 1;2 & 5;2 \end{pmatrix}$. Thus the game does not have an iteratively not strongly dominated equilibrium.

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Answer: no strictly dominant equilibrium. The procedure gives $\begin{pmatrix} 2;4 & 1;4 & 4;3 \\ 1;1 & 1;2 & 5;2 \end{pmatrix}$. Thus the game does not have an iteratively not strongly dominated equilibrium. Nash equilibria: $(1,1),(1,2),(2,2)$ and $(2,3)$.

Example

Example

Determine the strictly dominant equilibria, the iteratively not strongly dominated equilibria and the Nash equilibria of the game

$$\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}$$

Answer:

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Example

Determine the strictly dominant equilibria, the iteratively not strongly dominated equilibria and the Nash equilibria of the game

$$\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}$$

Answer: No player has as strictly dominant strategy, thus the game does not have a strictly dominant equilibrium. The procedure of iterative elimination of strongly dominated strategies gives the bimatrix $(5;2)$. Thus the game has an iteratively not strongly dominated equilibrium: $(3,3)$. The game has one Nash equilibrium: $(3,3)$.

Solution concepts (ctd.)

Theorem

1. *Each strictly dominant equilibrium is an iteratively not strongly dominated equilibrium.*

And if the game is finite:

1. *An iteratively not strongly dominated equilibrium is a unique Nash equilibrium.*

Solution concepts (ctd.)

Theorem

1. *Each strictly dominant equilibrium is an iteratively not strongly dominated equilibrium.*

And if the game is finite:

1. *An iteratively not strongly dominated equilibrium is a unique Nash equilibrium.*
2. *Each Nash equilibrium is an iteratively not strongly dominated strategy profile. (So each Nash equilibrium survives the procedure.)*

Proof.

1. Already in first steps of procedure all strategies are removed with the exception of strictly dominant ones.
2, 3. One verifies that in each step of the procedure the set of Nash equilibria remains the same. (See the text book.) □

Nash equilibria in the continuous case

So in this lesson we also deal with games, like the Cournot oligopoly, where each player has infinitely many strategies. For such games one need calculus in order to determine various fundamental objects, like Nash equilibria.

Concerning Nash equilibria: sometimes (in economics even 'often') they are the solutions of the n equations

$$\frac{\partial f_i}{\partial x_i} = 0 \ (i = 1, \ldots, n)$$

in $x_1, \ldots, x_n$.

Mixed strategies

Some games do not have a Nash equilibrium. However such a game may have a Nash equilibrium, if one plays the strategies with probabilities. Such a strategy is called a mixed strategy; up to now we dealt with pure strategies . More precisely

- Mixed strategy of player i: probability density over his strategy set X_i.

With mixed strategies, payoffs have the interpretation of expected payoffs. As the mixed strategy variant of a game in strategic form again is a game in strategic form, all above introduced fundamental notions, as that of Nash equilibrium, also make sense in the context of mixed strategies.

An important result is: each Nash equilibrium is a Nash equilibrium in mixed strategies. (See text book for formal proof.)

Bimatrix game with mixed strategies

Consider a 2×2 bimatrix game

$$(A; B)$$

(A concerns the part of the bimatrix for the row player and B the part for the column player.)

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Strategies: $(p, 1-p)$ for player 1 and $(q, 1-q)$ for player B. This means player 1 plays row 1 with probability p and row 2 with probability $1-p$. And player 2 plays column 1 with probability q and column 2 with probability $1-q$.

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Expected payoffs:

$$\bar{f}_1(p,q) = (p, 1-p) * A * \begin{pmatrix} q \\ 1-q \end{pmatrix},$$

$$\bar{f}_2(p,q) = (p, 1-p) * B * \begin{pmatrix} q \\ 1-q \end{pmatrix}.$$

Example

Determine the Nash equilibria in pure strategies and the Nash equilibria in mixed strategies for

$$\begin{pmatrix} 0; 0 & 1; -1 \\ 2; -2 & -1; 1 \end{pmatrix}.$$

Answer:

Example

Determine the Nash equilibria in pure strategies and the Nash equilibria in mixed strategies for

$$\begin{pmatrix} 0;0 & 1;-1 \\ 2;-2 & -1;1 \end{pmatrix}.$$

Answer: No Nash equilibria in pure strategies.

$$\bar{f}_1(p;q) = (p, 1-p) * A * \begin{pmatrix} q \\ 1-q \end{pmatrix} = (-4q+2)p + 3q - 1,$$

$$\bar{f}_2(p;q) = (p, 1-p) * B * \begin{pmatrix} q \\ 1-q \end{pmatrix} = (4p-3)q + 1 - 2p.$$

Example (ctd.)

Solving $\frac{\partial \bar{f}_1}{\partial p} = -4q + 2 = 0$ and $\frac{\partial \bar{f}_2}{\partial q} = 4p - 3 = 0$ (for the Nash equilibria in mixed strategies which are not pure Nash equilibria) gives the Nash equilibrium in mixed strategies

$$p = 3/4, q = 1/2.$$

As there is no Nash equilibrium in pure strategies, the conclusion is that there $p = 3/4,\ q = 1/2$ is the unique Nash equilibrium in mixed strategies.

If You like also see `https://www.youtube.com/watch?v=pC--lK8KNwo` for period 14:18-19:54. There another way is described how one can find the mixed Nash equilibria. However, this way only simplifies for very simple bimatrix games.

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Answer: No Nash equilibria in pure strategies. Nash equilibrium in mixed strategies: $p = q = 1/2$.

Existence of Nash equilibria

Theorem

(Nikaido-Isoda.) Each game in strategic form where

1. *each strategy set is a convex compact subset of some* $\mathbb{R}^n$,
2. *each payoff function is continuous,*
3. *each conditional payoff function is quasi-concave,*

has a Nash equilibrium.

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Proof.

This is a deep theoretical result. A proof can be based on Brouwer's fixed point theorem. See text book for the proof of a simpler case (Theorem 7.2., i.e. the next theorem). □

Theorem of Nash

Theorem

Each bi-matrix-game has a Nash equilibrium in mixed strategies.

Proof.

Apply the Nikaido-Isoda result. □

Antagonistic game

Consider an antagonistic game : two players and $f_1 + f_2 = 0$. (Cfr. with Exercise 7.7 in the text book.)

Theorem

If (a_1, a_2) and (b_1, b_2) are Nash equilibria, then $f_1(a_1, a_2) = f_1(b_1, b_2)$ and $f_2(a_1, a_2) = f_2(b_1, b_2)$.

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Proof.

$f_1(a_1, a_2) \geq f_1(b_1, a_2) = -f_2(b_1, a_2) \geq -f_2(b_1, b_2) = f_1(b_1, b_2)$.
In the same way $f_1(b_1, b_2) \geq f_1(a_1, a_2)$. Therefore $f_1(a_1, a_2) = f_1(b_1, b_2)$ and thus $f_2(a_1, a_2) = f_2(b_1, b_2)$. □

Little test

Are the following statements about games in strategic form true or false?

a. If each player has a dominant strategy, then there exists a unique Nash equilibrium.
b. A player has at most one strictly dominant strategy.
c. The 2×2-bi-matrix-game:

$$\begin{pmatrix} 4;0 & 2;-2 \\ 0;1 & 1;0 \end{pmatrix}$$

has a strictly dominant equilibrium.

Little test (ctd.)

d. The 3×2-bi-matrix-game:

$$\begin{pmatrix} 4;0 & 2;-2 \\ 0;1 & 1;0 \\ 2;-1 & 3;-2 \end{pmatrix}$$

does not have a Nash equilibrium in mixed strategies.

e. If each strategy profile is a Nash equilibrium, then each payoff function is constant.

f. Each fully cooperative strategy profile is Pareto efficient.

g. In a zero-sum game each strategy profile is Pareto efficient.

Little test (ctd.)

h. It is possible that a pure strategy is not strongly dominated by a pure strategy, but is by a mixed strategy.
i. It is possible that a best-reply-correspondence of a player is empty-valued, i.e. that given strategies of the other players there does not exist a best reply of that player.

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i. It is possible that a best-reply-correspondence of a player is empty-valued, i.e. that given strategies of the other players there does not exist a best reply of that player.

Answer:

Little test (ctd.)

h. It is possible that a pure strategy is not strongly dominated by a pure strategy, but is by a mixed strategy.

i. It is possible that a best-reply-correspondence of a player is empty-valued, i.e. that given strategies of the other players there does not exist a best reply of that player.

Answer: aF bT cT dF eF fT gT hT iT