Advanced Microeconomics: Game Theory

Lesson 2: Motivation and Outlook

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What You will learn

After studying Lesson 2, You

- should be able to explain what game theory is about;
- should be familiar with various specific games;
- should know which real-world types of games one distinguishes.

What is game theory?

Traditional game theory deals with mathematical models of conflict and cooperation in the real world between at least two rational intelligent players.

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- Player: humans, organisations, nations, animals, computers,...
- Situations with one player are studied by the classical optimisation theory.
- ‘Traditional’ because of rationality assumption.

Nature of game theory

- Applications: parlour games, military strategy, computer games, biology, economics, sociology, psychology anthropology, politocology.
- Game theory provides a language that is very appropriate for conceptual thinking.
- Many game theoretical concepts can be understood without advanced mathematics.
- Aim of game theory is to understand/predict how games will be played.

Outcomes and payoffs

- A game can have different outcomes. Each outcome has its own payoffs for every player.
- Nature of payoff: money, honour, activity, nothing at all, utility, real number, ...
- Interpretation of payoff: ‘satisfaction’ at end of game.
- In general it does not make sense to speak about ‘winners’ and ‘losers’.

Rationality and intelligence

- Because there is more than one player, especially rationality becomes a problematic notion. Here is a simple try: a player is rational if he has well-defined preferences concerning the outcomes of the game.
- Intelligence also is a not so easy notion. It presupposes an intelligent player and refers to the (rational) goal of that player. Intelligence has to do with the way the goal is approached.
- So rationality' and 'intelligence' are different concepts and the intelligence notion presupposes which type of rationality we are speaking about.

Rationality and intelligence

What would You as player 1 play in the following bimatrix game:

$$\begin{pmatrix} -1;-1 & -3;0 \\ 0;-3 & -2;-2 \end{pmatrix};$$

row 1 or row 2?

This game is the classical prisoner's dilemma game (of A. Tucker).

Now let us consider various concrete games. These games will be used in order to illustrate the abstract theory that we develop in the next lessons. These games concern:

- Tic-tac-toe.
- Hex.
- Hotelling bimatrix game.
- Cournot oligopoly (will be defined in Lesson 3).

Tic-tac-toe

- Here You can play this well-known game online: `https://playtictactoe.org/`.
- Many outcomes (more than three). However, only three types of outcomes: player 1 wins, draw, player 1 loses.

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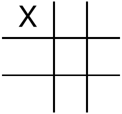
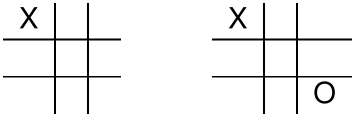
- Many outcomes (more than three). However, only three types of outcomes: player 1 wins, draw, player 1 loses.
- Example of a play of this game:

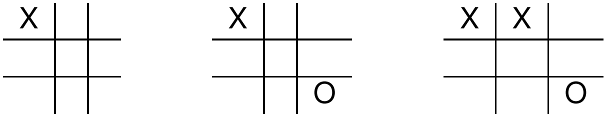
Tic-tac-toe (ctd.)



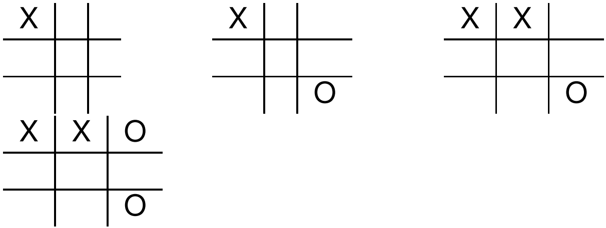
Tic-tac-toe (ctd.)



Tic-tac-toe (ctd.)



Tic-tac-toe (ctd.)



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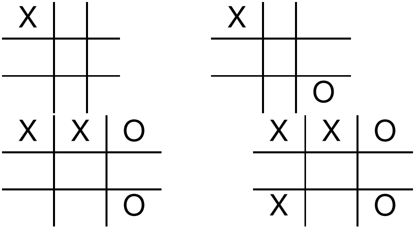


Tic-tac-toe (ctd.)

X		

X		
		O

X	X	
		O

X	X	O
		O

X	X	O
X		O

X	X	O
		O
X		O

Tic-tac-toe (ctd.)

X		

X		
		O

X	X	
		O

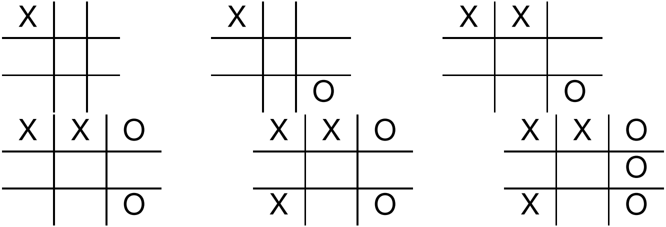
X	X	O
		O

X	X	O
X		O

X	X	O
		O
X		O

So: player 2 is the winner.

Tic-tac-toe (ctd.)



So: player 2 is the winner.

Question: Is player 1 intelligent? Is player 1 rational?

Tic-tac-toe (ctd.)

X		

X		
		O

X	X	
		O

X	X	O
		O

X	X	O
X		O

X	X	O
		O
X		O

So: player 2 is the winner.

Question: Is player 1 intelligent? Is player 1 rational?
Answer:

Tic-tac-toe (ctd.)

X		

X		
		O

X	X	
		O

X	X	O
		O

X	X	O
X		O

X	X	O
		O
X		O

So: player 2 is the winner.

Question: Is player 1 intelligent? Is player 1 rational?
Answer: We do not know. However, if player 1 is rational, then he is not intelligent.

Hex

Please see
`http://www.lutanho.net/play/hex.html`
and play this game several times.

1. The game at the above web page has an 11×11 board. In fact one can play the hex game also for other sizes of the board.
2. Hex was invented independently by Piet Hein and John Nash.
3. Hex can not end in a draw. ('Equivalent' with Brouwer's fixed point theorem in two dimensions.)
4. We shall prove later after having developed some theory that player 1 always can win the game.
5. If You can give a winning strategy for hex, then You solved one of the six '1-million-dollar problems'.

Hotelling bimatrix game

The Hotelling bi-matrix game depends on a parameter n being a positive integer and a parameter w with $0 < w \leq 1$. Consider the $n + 1$ points of $H := \{0, 1, \ldots, n\}$ on the real line, to be referred to as *vertices*.

Two players simultaneously and independently choose a vertex. If player 1 (2) chooses vertex x_1 (x_2), then the 'hinterland' of a player is the set of vertices that is the closest to his choice; a vertex that has equal distance to both players, a so-called 'shared vertex', belongs to both hinterlands.

In order to define the payoffs resulting from the choices x_1 and x_2, it is good to first consider the case $w = 1$ (dealing with so-called inelastic demand).

Hotelling bimatrix game

So suppose $w = 1$. Then the payoff $f_i(x_1, x_2)$ of player i is the number of non-shared vertices in his hinterland and half times the number of shared vertices in his hinterland.

Now consider the general case where $0 < w \leq 1$; the case where $0 < w < 1$ is called 'elastic demand'. The payoff $f_i(x_1, x_2)$ is calculated as follows: a non-shared vertex in the hinterland of player i at distance d to x_i contributes w^d to his payoff and a shared vertex contributes $w^d/2$.

Hotelling bimatrix game; $n = 7$ and $w = 1$

Action profile (5,2) :

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Action profile (5,2) :

Payoffs:

$1 + 1 + 1 + 1 = 4$

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Action profile (0,3) :

Hotelling bimatrix game; $n = 7$ and $w = 1$

Action profile (5,2) :

Payoffs:

$1 + 1 + 1 + 1 = 4$

$1 + 1 + 1 + 1 = 4$

Action profile (0,3) :

Payoffs

$1 + 1 = 2$

$1 + 1 + 1 + 1 + 1 + 1 = 6$

Hotelling bimatrix game; $n = 7$ and $w = 1$ (ctd.)

Action profile (2,6) :

Hotelling bimatrix game; $n = 7$ and $w = 1$ (ctd.)

Action profile (2,6) :

Payoffs:

$1 + 1 + 1 + 1 + \frac{1}{2} = 4\frac{1}{2}$

$\frac{1}{2} + 1 + 1 + 1 = 3\frac{1}{2}$

Hotelling bimatrix game; $n = 7$ and $w = 1$ (ctd.)

Action profile (2,6) :

Payoffs:

$1 + 1 + 1 + 1 + \frac{1}{2} = 4\frac{1}{2}$

$\frac{1}{2} + 1 + 1 + 1 = 3\frac{1}{2}$

Action profile (3,3) :

Hotelling bimatrix game; $n = 7$ and $w = 1$ (ctd.)

Action profile ($\color{blue}{2}$,$\color{red}{6}$) :

Payoffs:

$\color{blue}{1} + \color{blue}{1} + \color{blue}{1} + \color{blue}{1} + \color{blue}{\tfrac{1}{2}} = \color{blue}{4\tfrac{1}{2}}$

$\color{red}{\tfrac{1}{2}} + \color{red}{1} + \color{red}{1} + \color{red}{1} = \color{red}{3\tfrac{1}{2}}$

Action profile ($\color{blue}{3}$,$\color{red}{3}$) :

Payoffs:

$\color{blue}{\tfrac{1}{2}} + \color{blue}{\tfrac{1}{2}} + \color{blue}{\tfrac{1}{2}} + \color{blue}{\tfrac{1}{2}} + \color{blue}{\tfrac{1}{2}} + \color{blue}{\tfrac{1}{2}} + \color{blue}{\tfrac{1}{2}} + \color{blue}{\tfrac{1}{2}} = \color{blue}{4}$

$\color{red}{\tfrac{1}{2}} + \color{red}{\tfrac{1}{2}} + \color{red}{\tfrac{1}{2}} + \color{red}{\tfrac{1}{2}} + \color{red}{\tfrac{1}{2}} + \color{red}{\tfrac{1}{2}} + \color{red}{\tfrac{1}{2}} + \color{red}{\tfrac{1}{2}} = \color{red}{4}$

Hotelling bimatrix game; $n = 5$ and $w = 1/4$

Action profile (1,3) :

Hotelling bimatrix game; $n = 5$ and $w = 1/4$

Action profile (1,3) :

Payoffs:

$\frac{1}{4} + 1 + \frac{1}{8} = 1\frac{3}{8}$

$\frac{1}{8} + 1 + \frac{1}{4} + \frac{1}{16} = 1\frac{7}{16}$

Action profile (1,4) :

Hotelling bimatrix game; $n = 5$ and $w = 1/4$

Action profile (1,3) :

Payoffs:

$\frac{1}{4} + 1 + \frac{1}{8} = 1\frac{3}{8}$

$\frac{1}{8} + 1 + \frac{1}{4} + \frac{1}{16} = 1\frac{7}{16}$

Action profile (1,4) :

Payoffs:

$\frac{1}{4} + 1 + \frac{1}{4} = 1\frac{1}{2}$

$\frac{1}{4} + 1 + \frac{1}{4} = 1\frac{1}{2}$

Hotelling bimatrix game; $n = 7$ and $w = 1/2$

Action profile (2,6) :

Hotelling bimatrix game; $n = 7$ and $w = 1/2$

Action profile (2,6) :

Payoffs:

$\frac{1}{4} + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{8} = \frac{11}{8}$

$\frac{1}{8} + \frac{1}{2} + 1 + \frac{1}{2} = \frac{17}{8}$

Real-world types

In order to set up a theory for games one has to specify how the games that one considers relate to the real-word. (In red what we will assume always later when we develop the theory.)

- all players are rational – players may be not rational

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- all players are intelligent – players who may be not intelligent
- binding agreements – no binding agreements

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- binding agreements – no binding agreements
- chance moves – no chance moves

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- chance moves – no chance moves
- communication – no communication

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- chance moves – no chance moves
- communication – no communication
- static game – dynamic game

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- all players are intelligent – players who may be not intelligent
- binding agreements – no binding agreements
- chance moves – no chance moves
- communication – no communication
- static game – dynamic game
- transferable payoffs – no transferable payoffs

Real-world types (ctd.)

- interconnected games – isolated games
- perfect information – imperfect information
- complete information – incomplete information

The choices we made in red are very appropriate for dealing with non-cooperative game theory. Cooperative game theory will not be dealt with in this course.

Perfect information

- A player has perfect information if he knows at each moment when it is his turn to move how the game was played untill that moment.

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- A game is with (im)perfect information if (not) all players have perfect information.
- Chance moves are compatible with perfect information.
- Examples of games with perfect information: tic-tac-toe, chess, ...
  Examples of games with imperfect information: poker, monopoly (because of the cards, not because of the die).

Complete information

- A player has complete information if he knows all payoff functions.

Complete information

- A player has complete information if he knows all payoff functions.
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Complete information

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- A player has incomplete information if he does not have complete information.
- A game is with (in)complete information if (not) all players have complete information.
- Examples of games with complete information: tic-tac-toe, chess, poker, monopoly, ...
  Examples of games with incomplete information: auctions, oligopoly models where firms only know the own cost functions, ...

Common knowledge

Something is common knowledge if everybody knows it

Common knowledge

Something is common knowledge if everybody knows it and in addition that everybody knows that everybody knows it

Common knowledge

Something is common knowledge if everybody knows it and in addition that everybody knows that everybody knows it and in addition that everybody knows that everybody knows that everybody knows it and ...

Common knowledge

A group of dwarfs with red and green caps are sitting in a circle around their king who has a bell. In this group it is common knowledge that every body is intelligent. They do not communicate with each other and each dwarf can only see the color of the caps of the others, but does not know the color of the own cap. The king says: ”Here is at least one dwarf with a red cap.”. Next he says: “I will ring the bell several times. Those who know their cap color should stand up when i ring the bell.”. Then the king does what he announced.

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Common knowledge (ctd.)

The spectacular thing is that there is a moment where a dwarf stands up. Even, when there are M dwarfs with red caps that all these dwarfs simultaneously stand up when the king rings the bell for the M-th time.

Do not worry if You do not see why this claim this true. Dealing with such things is very advanced and quite important for the fundamental basis of game theory. However, it is to advanced for our (relatively simple) Advanced Microeconomics course. But if You like to know why the claim is true, then please have a look at

`https://en.wikipedia.org/wiki/Common_knowledge_(logic)`

where a similar situation is dealt with.

Appetizer

The time has come to start with developing theory. This will happen in Lessons 3, 4, 5 and 6.

Among other things we shall: show that tic-tac-toe, hex, chess, checkers and various other (parlor) games have a so-called value, in particular prove the claims made about the hex game and explain what can be said about how games without value will be played, like the Hotelling bimatrix game, Cournot oligopoly and many other economic games