

Advanced Microeconomics

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Exercises 3

Exercise 1 Consider a game in strategic form with two players. The game is called ‘strictly competitive’ if for all strategy profiles (x_1, x_2) and (y_1, y_2)

$$f_1(x_1, x_2) \geq f_1(y_1, y_2) \Leftrightarrow f_2(x_1, x_2) \leq f_2(y_1, y_2).$$

The game is called a ‘constant-sum game’ if there exists a number $c \in \mathbb{R}$ such that for each strategy profile (x_1, x_2) it holds that $f_1(x_1, x_2) + f_2(x_1, x_2) = c$.

- Show that each constant-sum game is strictly competitive.
- Show that in a strictly competitive game each strategy profile is strongly Pareto efficient.

Exercise 2 Make Exercise 7.4 (on strong domination) from the text book.

Exercise 3 Given the following bimatrix game:

$$\begin{pmatrix} 3; 8 & 4; 8 & 2; 3 \\ 1; 7 & 2; 6 & 8; 1 \\ 3; 4 & 4; 4 & 2; 2 \\ 1; 1 & 1; -1 & 1; -1 \end{pmatrix}.$$

- Determine the best reply correspondences.
- Determine the dominant strategies and the strictly dominant strategies.
- Determine the strategy profiles that survive the procedure of elimination of strongly dominated strategies.
- Determine the Nash equilibria.
- Determine the strongly Pareto-efficient strategy profiles and the weakly Pareto-efficient strategy profiles.

Exercise 4 Consider the bimatrix game

$$\begin{pmatrix} 5; 12 & 0; 0 \\ 0; 0 & 10; 4 \end{pmatrix}.$$

- Determine, in case of pure strategies, the best reply correspondences, and if it/they exist, the strictly dominant Nash equilibrium, the iterative not strongly dominated equilibrium and the Nash equilibria.
- Determine, in case of mixed strategies, the best reply correspondences and the Nash equilibria.

Exercise 5 An important example of a game in strategic form is a homogeneous cournot-oligopoly, by which we understand in this exercise a game in strategic form where each player i has as strategy set $X_i = [0, m_i]$ with $m_i > 0$ and a payoff function f_i of the form

$$f_i(x_1, \dots, x_n) = p(x_1 + \dots + x_n)x_i - c_i(x_i),$$

where $p : [0, m_1 + \dots + m_n] \rightarrow \mathbb{R}$ and $c_i : X_i \rightarrow \mathbb{R}$. Here x_i is called ‘production level’ of firm i . f_i is called ‘profit function’ of firm i , p ‘inverse demand function’ and c_i ‘cost function’ of firm i . A Nash equilibrium of this game also is called cournot equilibrium.

Suppose that p is decreasing, concave and twice differentiable and that each c_i is convex and twice differentiable.

- a. Prove that each conditional profit function is concave.
- b. Prove, using the Nikaido-Isoda theorem, that the game has at least one cournot equilibrium.

Short solutions.

Solution 1 a. For a constant-sum game we have $f_1 + f_2 = c$. So $f_1(a, b) \geq f_1(c, d) \Leftrightarrow c - f_2(a, b) \geq c - f_2(c, d) \Leftrightarrow f_2(a, b) \leq f_2(c, d)$. Thus the game is strictly competitive.

b. By contradiction suppose \mathbf{a} is strongly Pareto-inefficient. Then there exists a Pareto improvement \mathbf{b} of \mathbf{a} . So then there exists a player, say 1, who at \mathbf{b} has a greater payoff, and the other player does not have a smaller payoff. Thus $f_1(\mathbf{b}) > f_1(\mathbf{a})$ and $f_2(\mathbf{b}) \geq f_2(\mathbf{a})$. As the game is strictly competitive, we have $f_1(\mathbf{b}) \leq f_1(\mathbf{a})$, a contradiction.

Solution 2 We consider the bimatrix-game $(A; B)$ with $B = \begin{pmatrix} 0 & -3 & -4 \\ 4 & 5 & 8 \end{pmatrix}$. Then the second pure strategy of player 2 is not strongly dominated by a pure strategy. But for each mixed strategy (p_1, p_2) of player 1 we have $f_2((p_1, p_2), (1/2, 0, 1/2)) = 6 - 8p_1 > 5 - 8p_1 = f_2((p_1, p_2), (0, 1, 0))$. Therefore the second pure strategy of player 2 is strongly dominated by his mixed strategy $(1/2, 0, 1/2)$.

Solution 3 a. $R_1(1) = \{1, 3\}$, $R_1(2) = \{1, 3\}$, $R_1(3) = \{2\}$, $R_2(1) = \{1, 2\}$, $R_2(2) = \{1\}$, $R_2(3) = \{1, 2\}$, $R_2(4) = \{1\}$.

b. Dominant strategies for player 1: do not exist.

Dominant strategies for player 2: the first.

Strictly dominant strategies: do not exist.

c. Step 1:

$$\begin{pmatrix} 3; 8 & 4; 8 \\ 1; 7 & 2; 6 \\ 3; 4 & 4; 4 \end{pmatrix}.$$

Step 2:

$$\begin{pmatrix} 3; 8 & 4; 8 \\ 3; 4 & 4; 4 \end{pmatrix}.$$

d. They are $(1, 1)$ (i.e. row 1 and column 1), $(1, 2)$, $(3, 1)$, $(3, 2)$.

e. Strongly: $(1, 2)$ $(2, 3)$. Weakly: $(1, 1)$, $(1, 2)$ $(2, 3)$, $(3, 2)$.

Solution 4 a. $R_1(1) = \{1\}$, $R_1(2) = \{2\}$, $R_2(1) = \{1\}$, $R_2(2) = \{2\}$.

No strictly dominant equilibrium and no iterative not strongly dominated equilibrium.

Nash equilibria: $(1, 1)$ and $(2, 2)$.

b. $\bar{f}_1(p; q) = (15q - 10)p + 10 - 10q$ and $\bar{f}_2(p; q) = (16p - 4)q + 4 - 4p$.

Nash equilibrium: $p = 1/4, q = 2/3$; $p = 0, q = 0$; $p = 1, q = 1$.

Solution 5 a. Consider a player i . Fix a strategy of each other players. Let a be the sum of these strategies. For this situation the conditional payoff function is

$$g(x_i) = p(x_i + a)x_i - c_i(x_i).$$

Therefore

$$g''(x_i) = p''(x_i + a)x_i + 2p'(x_i + a) - c_i''(x_i) \leq 0.$$

Thus g is concave.

b. Having part a, apply the Nikaido-Isoda equilibrium existence result (from the slides).